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ERRATA.

Of the figures intended to illustrate Mr. PLAYFAIR's paper in the last volume of the Transactions, Fig. 3 is named by mistake Fig. 1. Fig. 1 was omitted by accident, and is to be found at the end of this Part of the Transactions.

Philosophical Transactions, 1811, page 230, line 13, *for* $8^h\ 35'\ 45''$ *read* $8^h\ 2'\ 5''$.

Page 151 of this volume of the Transactions, line 20, *for* half *read* double.

PHILOSOPHICAL

TRANSACTIONS.

XII. *Observations of a second Comet, with Remarks on its Construction.* *By* William Herschel, *LL. D. F. R. S.*

Read March 12, 1812.

As we have lately had two comets to observe at the same time, I have called that of which the following observations are given, the second. Its appearance has been so totally different from that of the first, that every particular relating to its construction becomes valuable; and notwithstanding the unfavourable state of the weather at this time of the year, I have been sufficiently successful to obtain a few good views of the phenomena which this comet has afforded.

A short detail of the observations, in the order of their relation to the different cometic appearances, is as follows:

The Body of the Comet.

January 1, 1812. I viewed the second comet with several of my telescopes, and found it to have a considerable nucleus surrounded with very faint chevelure.

Jan. 2. The comet had a large round nucleus within its faint nebulosity. Not seeing it very well defined, and of so large a diameter, I doubted whether it could be the body of the comet; but although it might be called very large when supposed to be of a planetary construction, it was much too small for the condensed light of a head; its diameter, by estimation not exceeding 5 or 6 seconds.

By way of comparing the two comets together I viewed them alternately. The first, within a nebulosity which in the form of a brilliant head was of great extent, had nothing resembling a nucleus: the light of this head was very gradually much brighter up to the very middle; its small planetary body being invisible. The second comet, on the contrary, although surrounded by a faint chevelure, seemed to be all nucleus; for the abrupt transition from the central light to that of the chevelure would not admit of the idea of a gradual condensation of nebulosity, such as I saw in the head of the first comet; but plainly pointed out that the nucleus and its chevelure were two distinct objects.

Jan. 8. The comet had a pretty well defined nucleus with very faint chevelure. When magnified 170 times the nucleus, though less bright, was rather better defined.

Jan. 18. Within a very faint chevelure I saw the nucleus as before.

Jan. 20. The air being uncommonly clear, I saw the body of the comet well defined; and as the moon was already so far advanced in its orbit as to render future opportunities of viewing the comet very improbable, I ascertained the magnitude of its body, with a very distinct 10 feet reflector, by the following three observations.

First with a low power, which gave a bright image of the nucleus, I kept my attention fixed upon its apparent size; then looking away from the telescope, I mentally reviewed the impression its appearance had made on the imagination, in order to see whether it was a faithful picture of the object; and by looking again into the telescope I was satisfied of the similitude.

In the next place I used a deeper magnifier, and alternately viewed and remembered the appearance of the nucleus. It was fainter with this power.

The third observation was made in the same manner with a magnifier of 170. This showed the nucleus of a larger diameter, but much less bright, and not so well defined.

The next morning, having recourse to my usual experiment with a set of globules, by viewing them at a given distance with the same telescope and eye-glasses, I found that one of them, on which I fixed, gave me, as nearly as could be estimated, the same magnitude with the first eye-glass, and was proportionally magnified by the second and third, with only this difference, that the highest power showed the globule with more distinctness than it did the nucleus; and by trigonometry the angle under which I saw the globule was found to be 5",2744.*

It will be necessary to mention that in the calculations belonging to this comet, I have used the elements of Mr. GAUS, with a small correction of the longitude of the perihelion, which I found would answer the end of giving the observed place with sufficient accuracy from the 1st of January to the

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* I prefer this method of ascertaining the small diameter of a faint object to measuring it with a micrometer, which requires light to show the wires, and a high magnifying power to give an image sufficiently large for mensuration; neither of which conditions the present comet would admit.

20th. These calculations may however be repeated, if hereafter we should obtain elements improved by additional observations, made with fixed instruments; but the result, I may venture to say, will not be materially different.

The distance of the comet from the earth, the 20th of January when its apparent diameter was determined, was 1,0867, the mean distance of the earth from the sun being 1; whence we deduce a very remarkable consequence, which is, that the real diameter of its nucleus cannot be less than 2637 miles.

The Chevelure of the Comet.

Instead of that bright appearance, which in the first comet has been considered as the head, there was about the nucleus of the second a faint whitish scattered light, which may be called its chevelure.

Jan. 1. Examining the chevelure of the comet with a 10 feet reflector, I found that it surrounded the nucleus, not in the form of a head consisting of gradually much condensed nebulosity, but had the appearance of a faint haziness, which although of some extent, was not much brighter near the nucleus than at a distance from it.

Jan. 2. I viewed the two comets alternately. The first could only be distinguished from a bright globular nebula by the scattered light of its tail, which was still 2° 20′ long. The second comet, on the contrary, had nothing in its appearance resembling such a nebula: it consisted merely of a nucleus, surrounded by a very faint chevelure; and had it not been for an extremely faint light in a direction opposite to the sun, it would hardly have been intitled to the name of a comet;

having rather the appearance of a planet seen through an atmosphere full of haziness.

Jan. 8. The chevelure consisted of so faint a light that, when magnified only 170 times, it was nearly lost.

Jan. 18. The chevelure was extremely faint and of very little extent.

Jan. 20. The light of the moon, which was up, would not admit of further accurate observations on the chevelure.

The Tail of the Comet.

Jan. 1. With a low magnifying power, I saw in the 10 feet reflector an extremely faint scattered light, in opposition to the sun, forming the tail of the comet. It reached from the centre of the double eye-glass half way toward the circumference.

Jan. 8. The narrow, very faint scattered light beyond the chevelure remains extended in the direction opposite the sun.

Jan. 18. I estimated the length of the tail by the proportion it bore to the diameter of the field of the eye-glass, which takes in 38′ 39″, and found that it filled about one quarter of it, which gives 9′ 40″.

Jan. 20. On account of moonlight the tail was no longer visible.

From the angle which it subtended in the last observation, it will be found that its length must have been about 659 thousand miles.

Remarks on the Construction of the Comet.

The method I have taken in my last paper of comparing together the phenomena of different comets appears to me most likely to throw some light upon a subject which still remains involved in great obscurity. When the comet of which the observations have been given in this paper is compared with the preceding one, it will be found to be extremely different. Its physical construction appears indeed to approach nearly to a planetary condition. In its magnitude it bears a considerable proportion to the size of the planets; the diameter of its nucleus being very nearly one-third of that of the earth.

The light by which we see it is probably also planetary; that is to say reflected from the sun. For were it of a phosphoric, self-luminous nature, we could hardly account for its little density: for instance, the very small body of the first comet, at the distance of 114 millions of miles from the earth bore a magnifying power of 600, and was even seen better with this than with a lower one;* whereas the second, notwithstanding its large size, and being only at the distance of 103 millions, had not light enough to bear conveniently to be magnified 107 times; but if we admit this nucleus to be opaque, like the bodies of the planets, and of a nature not to reflect much light, then its distance from the sun, which the 20th of January was above 174 millions of miles, will explain the cause of its feeble illumination.

That the nucleus of this comet was surrounded by an atmosphere appears from its chevelure, which, though faint, was of considerable extent; and the elasticity of this atmosphere may be inferred from the spherical figure of the chevelure, proved by its roundness and equal decrease of light at equal distances from the centre.

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* See Observations of the First Comet.

The transparency of the atmosphere is partly ascertained from our seeing the nucleus through it, but may also be inferred by analogy from an observation of the first comet. It will be remembered that an atmosphere of great transparency, which had been seen for a long time, was lost when the comet receded from the sun, by the subsidence of some nebulous matter not sufficiently rarified to enter the regions of the tail.* Now as the existence of this atmosphere, when it was no longer visible, might have been doubted, the luminous matter suspended in it, which had already 20 days obstructed our view of it, happened fortunately to be once more elevated the 9th of December, and thereby enabled us, from its transparency and capacity of sustaining luminous vapours, to ascertain the continuance of its existence. By analogy, therefore, we may surmise that the faint chevelure of the second comet consists also of the condensation of some remaining phosphoric matter, suspended in the lower regions, of an elastic, transparent fluid, extending probably far beyond the chevelure without our being able to perceive it.

We might ascribe the little extent and extreme faintness of the tail to the great perihelion distance of the comet, if it had not already been proved, by the comparative view which in my last paper has been taken of the two comets of 1807 and 1811, that the effect of the solar agency depends entirely upon the state of the nebulous matter, which the comet in its approach exposes to the action of the sun. Our last comet therefore had probably but little *unperihelioned* matter in its atmosphere.

The high consolidation of the matter contained in the second

* See Observations of the First Comet.

comet is also much supported by the different appearance of the two comets in the observation of the 2d of January. In order to judge of them properly, we must consider their situation with regard to the sun and the earth; the first comet was 192 millions of miles from the sun; the second only 164: the first was at the same time 262 millions from the earth: the second only 83; but notwithstanding the great disadvantage of being 28 millions of miles farther from the sun, and about 179 millions farther from the earth, the first comet had the luminous appearance of a brilliant head accompanied by a tail 45 millions of miles in length; whereas the second comet, so advantageously situated, had only a very faint chevelure about its large but faint nucleus, with a still fainter tail, whose length has been shown not much to exceed half a million.

If then the effect of the action of the sun on the comets at the time of their perihelion passage is more or less conspicuous, according to the quantity of unperihelioned nebulous matter they contain, we may by observation of cometic phenomena arrange these celestial bodies into a certain order of consolidation, from which, in the end, a considerable insight into their nature and destination may be obtained. The three last observed comets, for instance, will give us already the following results.

The comet of which this paper contains observations, is of such a construction that it was but little more affected by a perihelion passage than a planet would have been. This may be ascribed to its very advanced state of consolidation, and to its having but a small share of phosphoric or nebulous matter in its construction.

That of the year 1807 was more affected, and although considerably condensed, showed clearly that it conveyed a great quantity of nebulosity to the perihelion passage.

The comet of last year contained with little solidity a most abundant portion of nebulous matter, on which, in its approach to the perihelion, the action of the sun produced those beautiful phenomena, which have so favourably afforded an opportunity for critical observations.

XIII. *Additional Experiments on the Muriatic and Oxymuriatic Acids. By* William Henry, *M. D. F. R. S. V. P. of the Lit. and Phil. Society, and Physician to the Infirmary, at Manchester.*

Read March 19, 1812.

THE experiments, which form the subject of the following pages, are intended as supplementary to a more extensive series, which the Royal Society did me the honour to insert in their Transactions for the year 1800.* Of the general accuracy of those experiments, I have since had no reason to doubt; and their results, indeed, are coincident with those of subsequent writers of the highest authority in chemistry. My attention has been again drawn to the subject by the important controversy which has lately been carried on between Mr. Murray and Mr. John Davy, respecting the nature of muriatic and oxymuriatic acids;† and I have been induced, by some hints which the discussion has suggested, not only to repeat the principal experiments described in my memoir, but to institute others, with the advantage of a more perfect apparatus than I then possessed, and of greater experience in the management of these delicate processes.

This repetition of my former labours has discovered to me an instance, in which I have failed in drawing the proper conclusion from facts. In two comparative experiments on the electrization of equal quantities of muriatic acid gas, the one of which was dried by muriate of lime, and the other was in its natural state, I found a difference of not more than one *per cent.* in the hydrogen evolved, relatively to the original bulk of the gas.* Yet, notwithstanding these results, I have expressed myself inclined to believe that some water is abstracted by that deliquescent salt; and this belief was confirmed, several years afterwards, by the event of an experiment in which muriatic acid gas, dried by muriate of lime, gave only $\frac{1}{35}$ its bulk of hydrogen,† a proportion much below the usual average. The question, however, was too interesting to be left in any degree of uncertainty; and I have, therefore, made several fresh experiments with the view to its decision. In the course of these I have found, that though differences in the results are produced by causes apparently trivial, some of which I shall afterwards point out, yet that under equal circumstances, precisely the same relative proportion of hydrogene gas is obtained from muriatic acid gas, whether exposed or not to muriate of lime; and that its greatest amount does not exceed $\frac{1}{16}$ or $\frac{1}{14}$ the original volume of the acid gas.

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* Page 188.

† Nicholson's Journal, xxviii. and xxix.

In the paper last quoted,‡ I have also described an experiment, in which sensible heat was evolved by bringing muriate of lime into contact with muriatic acid gas; a fact which, if established, would go far to prove the existence of water in that gas. But on repeating the experiment with muriate of lime recently cooled from fusion, and over mercury carefully deprived of all moisture by boiling, I was not able to discover any increase of temperature, though a very sensible air thermometer was inclosed in the vessel containing the gas. The evolution of heat takes place, only when the muriate of lime has attracted moisture, either from the atmosphere or the mercury, and is then owing to a condensation of a part of the gas.

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* Page 191.

† Phil. Trans. 1809, page 433.

‡ Page 433 note.

Essentially, the changes produced by electrifying muriatic acid over mercury are those which I have stated; viz. a contraction of the volume of the gas, the formation of muriate of mercury (calomel), and the evolution of hydrogen. Recent experiments, also, have confirmed the accuracy of the observation,* that when a certain effect has been produced by electricity, nothing is gained by continuing the process; for neither is more hydrogen evolved, nor can the contraction of bulk be carried any farther.

I have lately applied, to experiments on muriatic acid, an apparatus which I used advantageously for the analysis of ammonia.† It consists of a spherical glass vessel, into which are hermetically sealed two small tubes containing platina wires, the points of which approach within the striking distance. To the globular part is attached a neck, which may be closed, as occasion requires, either by a glass stopper or by a metal cap and stop-cock. Into a vessel of this kind, I introduced 4½ cubic inches of muriatic acid gas, and passed through it 3000 discharges from a Leyden jar; at the close of the process, no traces of moisture could be perceived on the inner surface of the vessel, nor could I discover, on opening the stopper, that any change of bulk had taken place. After absorbing the unchanged muriatic acid gas by a small quantity of water, a volume of gas remained, in which there were present 100 measures (each equal to one grain of mercury) of oxymuriatic acid gas, and 140 measures of hydrogen. Two causes might, perhaps, contribute to diminish, in some degree, the proportion of the former. It was difficult to exclude from the apparatus, on admitting the muriatic acid gas into it, two or three very minute globules of mercury, which became tarnished during the experiment, exactly as they would have been by oxymuriatic acid; and a small portion of the latter gas was probably also taken up by the water employed to absorb the muriatic acid.

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* Phil. Trans. 1800. p. 192.

† Ibid. 1809.

With the intention of giving greater effect to the electricity, I repeated the experiment in a vessel capable of containing not more than 1400 grains of quicksilver (about .41 of a cubic inch), the neck of which, being only $\frac{1}{5}$ of an inch in diameter, was better calculated to shew any minute change in the volume of the gas. On removing the stopper, however, no change of volume was apparent. The hydrogen evolved, instead of being more than in the former experiment, equalled in bulk only 20 grains of mercury. The production of oxymuriatic acid was sufficiently evinced by its effect in tarnishing some very small globules of quicksilver, which adhered to the inside of the vessel; but the minuteness of the quantity frustrated an attempt to measure it. From subsequent experiments on similar quantities of gas, confined in the same apparatus, it appeared that the electrization in this last instance had been continued much longer than was necessary; and that an equal effect was produced by $\frac{1}{8}$ the number of electrical discharges.

In this way of making the experiment, the greatest proportion of hydrogen gas obtainable from muriatic acid amounted only to about $\frac{1}{70}$th, while, by electrization over quicksilver, $\frac{1}{16}$ or $\frac{1}{14}$ was generally evolved. It was evident, then, that the mercury had considerable influence over the results; and I found, by experiments with tubes of different diameters, that the larger the surface of the mercury exposed to the gas, the more rapid and complete was the change. Its action was greatly accelerated, also, by causing the electric discharge to strike from the conducting wire, sealed into the tube, to the mercury, which was probably thus raised into vapour; for in some instances, the whole of the inner surface of the glass was coated with sublimed calomel.

The only way, in which the mercury appeared to me likely to be efficient in this case, was by removing the oxymuriatic acid as fast as it was formed; for I have never found any mixture of this gas in the results of experiments on muriatic acid, when carried on over quicksilver. Upon any theory of the constitution of muriatic acid, it may be expected that when, in a mixture of that acid gas with hydrogen and oxymuriatic acid gases, the two latter come to bear a certain proportion to the former, they will be brought within the sphere of mutual agency, and will reproduce muriatic acid. This point appears, from my experiments, to be attained, when the hydrogen and oxymuriatic acid, taken together, have the proportion to the muriatic acid, of about 1 to 35. The amount of the change, therefore, which is capable of being effected on muriatic acid gas, electrified without the contact of mercury, is limited by the reaction of the evolved hydrogen and oxymuriatic acid gases on each other, whenever they compose a certain proportion of the mixture. This proportion being attained, we only, by continuing the electrization, work in a circle.

It may now be inquired, what is the limitation to the action of electricity on muriatic acid gas which is confined over mercury? In this case, it was suggested to me by Mr. DALTON, who favoured me with his presence at most of the experiments, that the evolved hydrogen might possibly in some way prevent the effect from being carried beyond a certain amount. Availing myself of this hint, I mixed 90 measures of hydrogen gas with 400 of muriatic acid gas in its ordinary state, and passed 900 discharges through the mixture. It soon became evident that the addition of the hydrogen had produced an important difference in the results of the experiment; for the surface of the mercury, over which the gas rested, was untarnished after some hundred explosions, and was scarcely changed at the close of the process. When the residuary gas, the volume of which remained unaltered, was analyzed, it was found to contain the same quantity of muriatic gas as at the outset, and neither more nor less hydrogen. To explain the event of this modification of the experiment, on the old theory, we may suppose that by the action of electricity a particle of water is decomposed, and that the atom of oxygen, forcibly repelled from that of hydrogen with which it was associated, finds another atom of hydrogen uninfluenced by the electric fluid, and within the sphere of its attraction. With this it unites, and recomposes water. On the theory of Sir H. DAVY, the same series of decompositions and recombinations may be assumed to take place between the oxymuriatic acid and hydrogen.*

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* I am aware that there is an apparent inconsistency in supposing changes of precisely an opposite kind to be effected by the same means. But instances are not wanting, in which the very same elements are brought into combination by electric discharges, and are again disunited by the same agency. As examples, it may be sufficient at present to state, that nitrous acid and nitrous gas are generated by the action of the electric spark on mixtures of oxygen and nitrogen gases; and that, by

It still, however, remains to be determined, what is the source of the hydrogene gas, which, in a limited proportion, is always evolved by the electrization of muriatic acid? Does it result from the decomposition of water, existing as an element of the gas; or from the disunion of the oxymuriatic acid and hydrogen, which, according to Sir H. Davy's view, compose muriatic acid? The limitation to its amount, which, it formerly appeared to me,* could only be accounted for by the complete destruction of the water contained in the gas, may now be equally well explained, on the principle which I have just pointed out. The fact, also, that no appreciable change of bulk is produced by the electrization of the muriatic acid, when the presence of mercury is excluded, is perhaps favourable to the new theory. For since equal measures of hydrogen and oxymuriatic acids afford muriatic acid without any condensation of volume, no alteration of bulk should result from the disunion of those elements; and the products should be equal measures of the same gases. The proportions, which I obtained (100 to 140) did not, it must be acknowledged, exactly correspond with the theory; but the difference was not greater, than might naturally be expected from the circumstances of the experiment. That equal measures of hydrogen and oxymuriatic acid are really evolved, appears to me to be proved by the agreement, which I have in several experiments remarked, between the hydrogen gas obtained,

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the same power, they are again resolved into their elements. If this were the proper place, it might, I think, be rendered probable by several arguments, that electricity, when thus applied, acts rather by mechanical collision, than by inducing a change in the electrical states of the elements of bodies.

* Phil. Trans. 1800, p. 200.

and the contraction of volume in muriatic acid electrified over mercury. Now the latter effect of the process can be explained on no other principle than the absorption of oxymuriatic acid by the quicksilver.

When muriatic acid and oxygen gases are electrified together over mercury, a gradual diminution ensues in their bulk,[*] and the mercury becomes tarnished, precisely as by the contact of oxymuriatic acid. I have lately examined the agency of this process on a considerable quantity of the two gases confined in a vessel, into which they were admitted after exhausting it by the air-pump. The phenomena, which in this way of making the experiment are extremely decisive and interesting, are the production of water and of oxymuriatic acid. The former, combining with a portion of the undecomposed muriatic acid, is deposited in drops upon the inner surface of the vessel, in the state of liquid muriatic acid. When the stop-cock, which confine the gases, is opened under mercury, a quantity of that metal rushes in, and has its surface instantly tarnished. Besides this test of the production of oxymuriatic acid, its presence is rendered unequivocal (after absorbing the undecomposed muriatic acid by a few drops of water), both by its smell, and by its effect in discharging the colour of litmus paper.[†]

These results, it will be found, may be reconciled with either theory. According to the one which has been commonly received, the oxygen unites with the real acid of muriatic gas, which becoming oxymuriatic acid, *deposits* water. On Sir H. DAVY's view, the oxygen unites with the hydrogen of the muriatic acid, and *composes* water, while the oxymuriatic acid is merely an educt. I am not aware of any refinement of the process, by which the value of these two explanations can be compared. Something, however, would be gained by a precise determination of the proportions, in which the two gases saturate each other. For since, on Sir H. DAVY's theory, muriatic acid contains half its volume of hydrogen gas, two measures of which are known to be saturated by one of oxygen, it follows that muriatic acid gas should be changed into oxymuriatic by one-fourth of its bulk of oxygen. According to GAY LUSSAC and THENARD,* three measures of muriatic acid should condense one of oxygen (or only one-third their bulk), and should form two measures of oxymuriatic acid. Hitherto, I have not been able to satisfy myself respecting the true proportions of oxygen and muriatic acid gases, that are capable of being united by electricity; for though I have made several experiments with this view, they have not agreed in yielding similar results. The condensation of a part of the undecomposed acid by the water, which is formed during the process, will, probably, indeed, always be an impediment to our learning these proportions exactly. The fact is chiefly of value, as it affords an example of the production of oxymuriatic acid under the simplest possible circumstances; and as it shews unequivocally that, under such circumstances, the visible appearance of moisture is a part of the phenomena.

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[*] Phil. Trans. 1800, p. 193.

[†] Those who wish to repeat this experiment need not be deterred by the apprehension of the labour attending it; for 3 or 400 discharges, from a Leyden jar of moderate size, are sufficient to occasion a distinct precipitation of moisture. When a mixture of oxygen and muriatic acid gases is even suffered to stand over mercury, a gradual contraction of volume takes place; the muriatic acid, if in proper proportion, entirely disappears; and calomel is deposited upon the surface of the glass vessel; but, in this case, there is no visible production of moisture.

Manchester, Jan. 6, 1812.

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* Mémoires d'Arcueil, ii. 217.

XVI. *Of the Attraction of such Solids as are terminated by Planes; and of Solids of greatest Attraction. By* Thomas Knight, *Esq. Communicated by Sir* H. Davy, *LL. D. Sec. R. S.*

Read March 19, 1812.

MATHEMATICIANS, in treating of the attraction of bodies, have confined their attention, almost entirely, to those solids which are bounded by continuous curve surfaces; and Mr. PLAYFAIR, if I do not mistake, is the only writer, who has given any example of that kind of inquiry, which is the chief object of the present paper. This learned mathematician has found expressions* for the action of a parallelopiped; and of an isosceles pyramid, with a rectangular base, on a point at its vertex; and observes, on occasion of the first mentioned problem, that what he has there done, "gives some hopes of "being able to determine generally the attraction of solids "bounded by any planes whatever."

It is this general problem, that I venture to attempt the solution of, in what follows: viz. *any solid, regular or irregular, terminated by plane surfaces, being given, to find, both in quantity and direction, its action, on a point, given in position, either within or without it.*

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* Ed. Trans. Vol. VI. p. 228 to 243. It is proper however to observe, that Mr. PLAYFAIR's expression, at p. 242, for the action of a parallelopiped, requires to have its sign changed; being, as it stands at present, negative, from the manner of correcting the fluent.

Nor has the matter any difficulty, as far as *theory*[*] *only* is concerned; although, *actually* to find the attraction, of a body of very complicated figure, may, no doubt, be exceedingly laborious and troublesome: for no one, I suppose, will conceive, that it can be done in any other manner, than by a previous partition into more simple forms, each of which must have its action found separately.

Having completed this part of my subject in the three first sections, I next apply the formulas, given in §. 1, to find the attraction of certain complex bodies, which, though not bounded by planes, have yet a natural connexion with the preceding part of the paper. Finally, the fifth section treats, pretty fully, of solids of greatest attraction, under various circumstances; and I do not know, that any one of the problems there given has been before considered by mathematicians; whilst, on the other hand, the results of former writers are easily derived as corollaries.

For the sake of perspicuity, I have divided the paper into propositions, and shall terminate this short introduction by expressing a hope, that I may not be chargeable with unnecessary prolixity.

§. I.

Of the Attraction of Planes bounded by right Lines.

As all such figures may be divided into triangles, it seems natural to begin with these.

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[*] It is usual, I think, with mathematicians, to consider a thing as done, when it can be pointed out how it *may be* done. Thus M. LAGRANGE, in his excellent work "*De la Résolution des Equations numériques,*" says (p. 43) "cette méthode ne laisse, "ce me semble, rien à desirer." where, of course, he can only mean, as far as relates to *theory*.

Prop. 1.

Let rvm, fig. 1, be a triangle, right angled at r, and pm a right line, perpendicular to the plane of the triangle, at the angular point m; it is required to find the attraction of the triangle, on the point p, both in quantity and direction.

Conceive a plane to pass through the point p, parallel to the plane of the triangle, and, in it, the lines pg, po, respectively parallel to rm, rv. The problem will be solved, if we find the actions of the triangle, in the directions of the three rectangular co-ordinates pm, pg, po.

Draw ks parallel to rv, and put $a =$ pm, $b =$ rm, $T =$ mk, $t =$ kq; then pq $= \sqrt{a^2 + T^2 + t^2}$. Let $r =$ tang. vmr, then ks $= r \times$ km.

The element of the plane at q is $\dot{T} \times \dot{t}$, and its action, on p, in the direction pq, is $\frac{\dot{T}\dot{t}}{a^2 + T^2 + t^2}$; by resolving which, and putting A, B, C for the actions of the triangle, in the directions pm, pg, po, we get

$$A = \iint \frac{a\dot{T}\dot{t}}{(a^2 + T^2 + t^2)^{\frac{3}{2}}};\ B = \iint \frac{T\dot{T}\dot{t}}{(a^2 + T^2 + t^2)^{\frac{3}{2}}};\ C = \iint \frac{\dot{T}t\dot{t}}{(a^2 + T^2 + t^2)^{\frac{3}{2}}};$$

in all which expressions, we must first take the fluent, with respect to t, from $t = o$, to $t = rT$; and afterwards, with respect to T, from $T = o$, to $T = b$. To begin with A,— a first operation gives

$$A = \int \frac{arT\dot{T}}{(a^2 + T^2)(a^2 + (1 + r^2)T^2)^{\frac{1}{2}}}$$

which, if we put $\beta^2 = 1 + r^2$, will be changed to

$$A = \int \frac{arT\dot{T}}{\beta(a^2 + T^2)\left(\frac{a^2}{\beta^2} + T^2\right)^{\frac{1}{2}}}.$$

Put $z^2 = \frac{a^2}{\beta^2} + T^2$, whence $T^2 = z^2 - \frac{a^2}{\beta^2}$, $T\dot{T} = z\dot{z}$: by substituting these values we get

$$A = \int \frac{a r \dot{z}}{\beta\left(a^2 - \frac{a^2}{\beta^2} - a^2 + z^2\right)} = (\text{because } \beta^2 - 1 = r^2) \int \frac{a r \dot{z}}{\beta\left(\frac{a^2}{\beta^2} r^2 + z^2\right)},$$

which, if we multiply both numerator and denominator by $\frac{\beta^2}{a^2 r^2}$, becomes $\int \frac{\frac{\beta}{ar}\dot{z}}{1 + \frac{\beta^2}{a^2 r^2} z^2} = \text{arc}\left(\text{tang.} = \frac{\beta}{ar} z\right)$, and, by putting for z and β their values, we have at last

$${}^{*}A = \text{arc}\left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{1+r^2}{a^2} b^2}\right) - \text{arc}\left(\text{tang.} = \frac{1}{r}\right).$$

In like manner, a first integration of B gives

$$B = \int \frac{r T^2 \dot{T}}{(a^2 + T^2)(a^2 + (1+r^2)T^2)^{\frac{1}{2}}} = \int \frac{r T^2 \dot{T}}{\sqrt{1+r^2}\,(a^2+T^2)\left(\frac{a^2}{1+r^2} + T^2\right)^{\frac{1}{2}}}$$

$$= \frac{r}{\sqrt{1+r^2}} \int \left\{ \frac{\dot{T}}{\left(\frac{a^2}{1+r^2} + T^2\right)^{\frac{1}{2}}} - \frac{a^2 \dot{T}}{(a^2+T^2)\left(\frac{a^2}{1+r^2} + T^2\right)^{\frac{1}{2}}} \right\}.$$

Put $T = a$ tang. ϖ, then $\dot{T} = a$ sect. ${}^2\varpi\dot{\varpi}$, $a^2 + T^2 = a^2$ sect. ${}^2\varpi$; by this means the last term under the sign of the fluent is changed to $\frac{\dot{\varpi}}{\left(\frac{1}{1+r^2} + \text{tang.}\,{}^2\varpi\right)^{\frac{1}{2}}} = -\frac{\sqrt{1+r^2}}{r} \times \frac{\dot{\overline{r\,\text{sin.}\,\varpi}}}{(1 + r^2\,\text{sin.}\,{}^2\varpi)^{\frac{1}{2}}}$; wherefore, observing that tang. $\varpi = \frac{T}{a}$, and consequently sin. $\varpi = \frac{T}{\sqrt{a^2+T^2}}$, we find at last

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[*] This quantity can be put under another form, which may be better in some cases. If we denote by b' the side rv of the triangle, $r = \frac{b'}{b}$, and

$$A = \text{arc}\left(\text{tang.} = \frac{b}{b'} \times \frac{\sqrt{a^2+b^2+b'^2}}{a}\right) - \text{arc}\left(\text{tang.} = \frac{b}{b'}\right).$$

$$B = \frac{r}{\sqrt{1+r^2}} L\left(\frac{b}{a} + \sqrt{\frac{1}{1+r^2} + \frac{b^2}{a^2}}\right) - L\left\{\frac{rb}{\sqrt{a^2+b^2}} + \sqrt{1 + \frac{r^2 b^2}{a^2+b^2}}\right\}$$

$$-\frac{r}{\sqrt{1+r^2}} L . \frac{1}{\sqrt{1+r^2}}.$$

We have yet to find the value of C; and at first we get

$$C = \int \frac{\dot{T}}{(a^2+T^2)^{\frac{1}{2}}} - \int \frac{\dot{T}}{(a^2+(1+r^2)T^2)^{\frac{1}{2}}} = \int \frac{\frac{\dot{T}}{a}}{\left(1 + \frac{T^2}{a^2}\right)^{\frac{1}{2}}} - \frac{\dot{T}}{\sqrt{1+r^2}}$$

$$\int \frac{\frac{\dot{T}}{a}}{\left(\frac{1}{1+r^2} + \frac{T^2}{a^2}\right)^{\frac{1}{2}}},$$

which again integrated, becomes

$$C = L\left(\frac{b}{a} + \sqrt{1 + \frac{b^2}{a^2}}\right) - \frac{1}{\sqrt{1+r^2}} L\left(\frac{b}{a} + \sqrt{\frac{1}{1+r^2} + \frac{b^2}{a^2}}\right)$$

$$+ \frac{1}{\sqrt{1+r^2}} L . \frac{1}{\sqrt{1+r^2}}.$$

The expressions we have thus arrived at, for the action of a right angled triangle, are of such continual use in the following propositions, that it will be convenient to represent them by some concise symbol; and as they are functions of *a*, *b*, and *r* we may put

$$A = \text{arc}\left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{1+r^2}{a^2} b^2}\right) - \text{arc}\left(\text{tang.} = \frac{1}{r}\right) =$$

$$\varphi(a, b, r) - \text{arc}\left(\text{tang.} = \frac{1}{r}\right).$$

$$B = \frac{r}{\sqrt{1+r^2}} L\left(\frac{\sqrt{1+r^2}\, b}{a} + \sqrt{1 + \frac{1+r^2}{a^2} b^2}\right) - L\left(\frac{rb + \sqrt{a^2+(1+r^2)b^2}}{\sqrt{a^2+b^2}}\right)$$

$$= \chi(a, b, r).$$

$$C = L\left(\frac{b}{a} + \sqrt{1 + \frac{b^2}{a^2}}\right) - \frac{1}{\sqrt{1+r^2}} L\left(\frac{\sqrt{1+r^2}\, b}{a} + \sqrt{1 + \frac{1+r^2\, b^2}{a^2}}\right)$$

$$= \psi(a, b, r).$$

Cor. 1. If, whilst *r* remains constant, *b* and *a* are supposed to vary, but so as to preserve the same ratio to each other, the partial forces A, B, C will remain unchanged, as, consequently, will the total force, both in quantity and direction.

For, if we put $m \times a$ for b, the expressions of the forces become $\varphi(1, m, r) - \text{arc}\left(\text{tang.} = \frac{1}{r}\right), \chi(1, m, r), \psi(1, m, r)$; which are independent of the *absolute* values of a and b. It is scarcely necessary to observe, that arc $\left(\text{tang.} = \frac{1}{r}\right)$ is the arc, to the radius unity, corresponding to the angle rvm.

Cor. 2. When r becomes infinite, the triangle rmv is changed into a parallelogram, infinitely extended in the direction rv; in which case, the expressions of the forces become very simple, viz. $A = \text{arc}\left(\text{tang.} = \frac{b}{a}\right)$, $B = L.\frac{\sqrt{a^2+b^2}}{a}$, $C = L.\frac{b+\sqrt{a^2+b^2}}{a}$.

Prop. 2.

Let vmu, fig. 2, be any triangle whatever, pm a line perpendicular to the plane of the triangle, at the angular point m: from whence, let fall the perpendicular mr on the opposite side uv; moreover, let pg, po, be respectively parallel to mr, vu.

It is required to find the actions of the triangle vmu on the point p, in the directions pm, pg, po.

If we keep the same denominations as before, and put, besides, r′ = tang. umr, it is plain from the last proposition, and because the action of the whole must necessarily equal the sum of the actions of its parts, that

$$A = \varphi(a,b,r) + \varphi(a,b,r') - \text{arc}\left(\text{tang.} = \frac{1}{r}\right) - \text{arc}\left(\text{tang.} = \frac{1}{r'}\right);$$

$$B = \chi(a,b,r) + \chi(a,b,r'); \quad C = \psi(a,b,r) - (a,b,r').$$

When umv is a right angle, we shall evidently have arc

$\left(\text{tang.} = \frac{1}{r}\right) + \text{arc}\left(\text{tang.} = \frac{1}{r'}\right) = \frac{\pi}{2}$, π being the number 3,1415, &c.: this makes the expression for A somewhat simpler, in that case.

If it is the triangle vmu′ whose attraction we seek, we have, putting $r' = \text{tang. } u'mr$,

$$A = \varphi(a,b,r) - \varphi(a,b,r') - \text{arc}\left(\text{tang.} = \frac{1}{r}\right) + \text{arc}\left(\text{tang.} = \frac{1}{r'}\right);$$

$$B = \chi(a,b,r) - \chi(a,b,r'); \ C = \psi(a,b,r) - \psi(a,b,r').$$

Cor. 1. As a rhombus may be divided, from its centre, into four equal triangles, like that in fig. 2, but right angled at m, the angle lying at the centre; if b represent a perpendicular from the centre of a rhombus on one of its sides, and r and r' the tangents of the angles, that this perpendicular makes, at the centre, with the semi-diameters of the figure, we shall have for the action of the rhombus, on a point situated perpendicularly over its centre, at the distance a,

$$A = 4\varphi(a,b,r) + 4\varphi(a,b,r') - 2\pi.$$

Cor. 2. As any plane, terminated by right lines, may be divided into triangles from a point within it, we may find, by means of this proposition, the attraction of such a plane, on a point above it, both in quantity and direction. Let, for example, uvu′v′u, fig. 6, be the plane, p the attracted point; let fall the perpendicular pm on the plane, and from m draw right lines to the angles u, v, u′, v′; the plane will thus be divided into triangles, situated, with respect to the point p, like that in the proposition.

The attraction may still be found, if the perpendicular should fall without the figure; as in

Prop. 3.

To find the attraction of a triangle umv, fig. 3, on a point p any how situated.

Let fall from p the perpendicular pm′, on the plane of the triangle; join m′m, m′u, m′v. Find, by the last Prop. the attractions of the triangles m′uv, m′um, m′vm, on the point p; and resolve them into others in the directions of any three rectangular co-ordinates: and, when thus resolved, let the actions of

$$\left\{\begin{matrix} \mathrm{m'uv} \\ \mathrm{m'um} \\ \mathrm{m'vm} \end{matrix}\right\} \begin{matrix}\text{in the directions of these} \\ \text{co-ordinates be}\end{matrix} \left\{\begin{matrix} A\ , B\ , C \\ A', B', C' \\ A'', B'', C'' \end{matrix}\right\}.$$

It is plain, that the actions of the triangle umv, on p, in the directions of the same co-ordinates, will be

$$A - A' - A''; \ B - B' - B''; \ C - C' - C''.$$

There may be other cases of this proposition, in which the triangle and point are placed differently, with respect to each other, from what I have represented in fig. 3; but the reader, who understands the case that has been considered, will have no difficulty in any other that may occur.

Though the preceding propositions contain every thing that is necessary, for finding the attraction, both in quantity and direction, of any plane bounded by right lines; yet there are some cases worthy of a particular notice: as

Prop. 4.

To find the attraction of a rectangle mrvr′, fig. 4, on a point p situated perpendicularly over one of its corners as m.

Draw pg, po, parallel to mr, mr′, the sides of the rectangle; put $b = \mathrm{rm}$, $b' = \mathrm{r'm}$, $r = \text{tang. rmv}$, $r' = \text{tang. r'mv}$: then, if

A, B, C represent the actions of the rectangle, in the directions pm, pg, po, we find, by means of Proposition 1,

$$A = \varphi(a, b, r) + \varphi(a, b', r') - \frac{\pi}{2};$$

$$B = \chi(a, b, r) + \psi(a, b', r'); \quad C = \psi(a, b, r) + \chi(a, b', r').$$

We may eliminate r and r', from these expressions, by means of their values $\frac{b'}{b}$ and $\frac{b}{b'}$; and thus we may put A under a very simple form. It becomes, at first,

$$A = \text{arc}\left(\text{tang.} = \frac{b}{b'} \times \frac{\sqrt{a^2+b^2+b'^2}}{a}\right) + \text{arc}\left(\text{tang.} = \frac{b'}{b} \times \frac{\sqrt{a^2+b^2+b'^2}}{a}\right) - \frac{\pi}{2}.$$

But, by trigonometry, α and β representing any angles whatever, tang. $(\alpha + \beta = \frac{\text{tang.}\,\alpha + \text{tang.}\,\beta}{1 - \text{tang.}\,\alpha \times \text{tang.}\,\beta}$: the application of which formula gives us, instead of the foregoing expression,

$$A = \text{arc}\left(\text{tang.} = \frac{-a}{bb'}\sqrt{a^2+b^2+b'^2}\right) - \frac{\pi}{2};$$

and this again is easily changed into the following form,

$$A = \text{arc tang.} = \frac{bb'}{a\sqrt{a^2+b^2+b'^2}}\Big),$$ which is easily perceived to be the same as Mr. PLAYFAIR's expression.

In a similar manner, might the expressions for B and C be simplified: but it is perhaps easier to find new forms, *ab initio*. Thus we may get B immediately, from the double integral $B = \iint \frac{T\dot{T}\dot{t}}{(a^2+T^2+t^2)^{\frac{3}{2}}}$, if the fluent, with respect to t, be taken from $t = o$ to $t = b'$; and, with respect to T, from $T = o$ to $T = b$. The first integration gives $B = \int \frac{b'T\dot{T}}{(a^2+T^2)(a^2+b'^2+T^2)^{\frac{1}{2}}}$; and, by the second,

$$B = L.\frac{\sqrt{a^2+b^2}+b'}{a} - L.\frac{\sqrt{a^2+b'^2+b^2}+b'}{\sqrt{a^2+b^2}}.$$

It is plain that, to find C, we need only change b into b', and the reverse, in the last expression, whence

$$C = L.\frac{\sqrt{a^2+b^2}+b}{a} - L.\frac{\sqrt{a^2+b^2+b'^2}+b}{\sqrt{a^2+b'^2}}.$$

Prop. 5.

To find the attraction of a regular polygon, on a point situated perpendicularly over its centre.

As this figure is composed of isosceles triangles, if we put b for the perpendicular from the centre on one of the sides, and r for the tangent of half the angle at the centre, subtended by one of those sides, we have, by Prop. 1, for a polygon of n sides,

$$A = 2n \text{ arc} \left(\text{tang.} = \frac{1}{r}\sqrt{1+\frac{1+r^2}{a^2}b^2}\right) - 2n \text{ arc} \left(\text{tang.} = \frac{1}{r}\right),$$

which, because the last term $= (n-2)\pi$, by Euclid 1.32. *Cor.* 1, is

$$A = 2n \text{ arc} \left(\text{tang.} = \frac{1}{r}\sqrt{1+\frac{1+r^2}{a^2}b^2}\right) - (n-2)\pi.$$

Prop. 6.

To find the attraction of a circle, on a point situated perpendicularly over its centre.

This is only a particular case of the last proposition, when n is infinitely great and r infinitely small.

It is easy to see, that the arc whose tangent is $\frac{1}{r}\sqrt{1+\frac{1+r^2}{a^2}b^2}$ will have for its cosine $\frac{ra}{\sqrt{a^2+b^2}}$, if we keep only the first power of r; consequently we may put it under the form $\frac{\pi}{2}$ — arc $\left(\text{sine} = \frac{ra}{\sqrt{a^2+b^2}}\right) = .\frac{\pi}{2} - \frac{ra}{\sqrt{a^2+b^2}}$, very nearly; this multiplied by $2n$ is $n\pi - \frac{2nra}{\sqrt{a^2+b^2}} = n\pi - \frac{2\pi a}{\sqrt{a^2+b^2}}$ by putting π for nr. If we substitute this value in the expression for a regular polygon, it becomes $A = 2\pi \left\{1 - \frac{a}{\sqrt{a^2+b^2}}\right\}$, the well known expression of NEWTON.

§. II.

Of the Attraction of Pyramids, and generally of any Solids whatever that are bounded by Planes.

The simplest case of the attraction of such bodies as we are to consider, is that of a pyramid with the attracted point at the vertex: and it fortunately happens, that on this simple case the action of any body whatever may be made to depend; which is the reason of my placing the general problem in this section, though I afterwards treat separately of prisms.

This facility, in the case of pyramids, results from what was shewn in *Cor.* 1. *Prop.* 1, viz. that if we put x for a and mx for b, in the functions $\varphi(a, b, r)$, $\chi(a, b, r)$, $\psi(a, b, r)$, they will become $\varphi(1, m, r)$, $\chi(1, m, r)$ $\psi(1, m, r)$, into which x does not enter.

Prop. 7.

Let figure 5 represent a pyramid with a triangular base umv, the vertex p of the solid being in a line pm, perpendicular to the triangle at the angular point m. It is required to find the action of the pyramid on a point at that vertex.

Draw the perpendicular mr, also pg, po parallel to mr, rv. Join pr, and let r′m′ be parallel to rm. Call pm′, x; r′m′, y; then $y = mx$, m being the tangent of the angle rpm. The attraction of a triangular section of the solid, made by a plane, passing through r′m′, parallel to the base, will be found by *Prop.* 2, if we put x for a, and mx for b; and is in the respective directions pm, pg, po.

$$A = \varphi(1, m, r) + \psi(1, m, r') - \text{arc}\left(\text{tang.} = \frac{1}{r}\right) - \text{arc}\left(\text{tang.} = \frac{1}{r'}\right);$$

$$B = \chi(1, m, r) + \chi(1, m, r'); \; C = \psi(1, m, r) - \psi(1, m, r').$$

If we multiply these by $\dot{x}$ and take the fluents, the actions of the pyramid are found to be

$$A = x\varphi(1, m, r) + x\varphi(1, m, r') - x\,\text{arc}\left(\text{tang.} = \frac{1}{r}\right) - x\,\text{arc}\left(\text{tang.} = \frac{1}{r'}\right);$$

$$B = x\chi(1, m, r) + x\chi(1, m, r'); \; C = x\psi(1, m, r) - x\psi(1, m, r').$$

If the pyramid, whose action we were seeking, had been that whose is u′mv, we must have used the other values of A, B, C given in *Prop.* 2.

Prop. 8.

Let fig. 6 represent any pyramid whatever, whose base uvu′v′u is terminated by right lines; to find its attraction, on a point at the vertex p, both in quantity and direction.

Let a perpendicular from p meet the base at m, and draw lines from this point to all the angles u, v, &c. of the base. It is plain, that the solid will thus be divided into such pyramids as were considered in the last proposition; so that the problem is already solved.

Cor. We may apply this to the pyramid whose base is a rhombus, and the vertex placed perpendicularly over its centre. By proceeding as in the proposition, it will be divided into four equal triangular pyramids; and, using for each of them the notation of *Prop.* 7, we have for the action of the whole rhomboidal pyramid, on a point at its vertex,

$$A = 4x\varphi\,(1, m, r) + 4x\varphi\,(1, m, r') - 2\pi x.$$

The other attractions evidently destroy each other.

It is not necessary, in the above proposition, that the perpendicular pm should fall within the base; if it falls without, we shall however have occasion for the following problem.

Prop. 9.

Let umv, fig. 3, be any triangle whatever, p a point any how situated with respect to it; join pm, pu, pv.* It is required to find the attraction of the oblique pyramid pumv, whose base is the triangle umv, on a point at the vertex p.

Let fall, from p, the perpendicular pm′ on the plane of the base umv, draw the lines m′m, m′u, m′v. Find, by *Prop.* 7, the attractions of the pyramids pm′uv, pm′um, pm′vm, whose bases are m′uv, m′um, m′vm, and their common vertex p.

Resolve these attractions into others in the directions of any rectangular co-ordinates, and when thus resolved let the actions of the pyramids

$$\left\{\begin{matrix}\text{pm}'\text{uv}\\ \text{pm}'\text{um}\\ \text{pm}'\text{vm}\end{matrix}\right\}\ \text{in the directions of these co-ordinates be}\ \left\{\begin{matrix}A\ ,\ B\ ,\ C\\ A',\ B',\ C'\\ A'',\ B'',\ C''\end{matrix}\right\}.$$

It is plain, that the actions of the pyramid pumv, on the point p, in the directions of the same co-ordinates, will be $A - A' - A''$; $B - B' - B''$; $C - C' - C''$.

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* I have not actually drawn the lines, to avoid confusion in the figure.

Prop. 10.

Let fig. 4 represent, in every respect, the same as it did in *Prop.* 4; join pr, pv, pr′; it is required to find the attraction of the square pyramid pmrvr′, on the point p at its vertex.

If we proceed as in *Prop.* 7, but make use of the expressions, for the action of the rectangle mrvr′, found in *Prop.* 4, and put x for a, mx for b, $m'x$ for b', (where $m =$ tang. rpm, $m' =$ tang. r′pm) there will result

$$A = x\varphi\,(1, m, r) + x\varphi\,(1, m', r') - \frac{\pi x}{2};$$

$B = x\chi\,(1, m, r) + x\psi\,(1, m', r')$; $C = x\psi\,(1, m, r) + x\chi\,(1, m', r')$.

But it will be better to make use of the more simple expressions, that were given in *Prop.* 4, by which means, we get

$$A = \text{arc}\left(\text{tang.} = \frac{mm'}{\sqrt{1+m^2+m'^2}}\right)x$$

$$B = \left\{L.\left(\sqrt{1+m'^2}+m'\right) - L.\frac{\sqrt{1+m'^2+m^2}+m'}{\sqrt{1+m^2}}\right\}x$$

$$C = \left\{L.\left(\sqrt{1+m^2}+m\right) - L.\frac{\sqrt{1+m^2+m'^2}+m}{\sqrt{1+m'^2}}\right\}x.$$

Prop. 11.

Let the base of the pyramid be a regular polygon; the vertex situated perpendicularly over the centre of the base; the attracted point at the vertex.

By making use of the expression in Proposition 5, putting x for a, and mx for b (m being the tangent of the angle, at the vertex, formed by the axis of the pyramid and a line drawn from the vertex to the middle of one of the sides of the base, we get

$$A = 2nx \text{ arc } (\text{tang.} = \tfrac{1}{r}\sqrt{1 + (1 + r^2) m^2}) - (n - 2)\pi x.$$

Hitherto, I have considered the action of a pyramid, only on a point at its vertex; the case which next presents itself, is that of the attraction on a point p′ (fig. 5 and 6) any where in the produced axis mpp′. It would be easy to give a direct solution to this problem, but I choose rather to make it depend on the propositions that have been already established: and to shew that the functions $\varphi\,(a, b, r)$ $\chi\,(a, b, r)$ $\psi\,(a, b, r)$, of which so much use has been made, in the preceding investigations, are sufficient in all cases of the attraction of bodies bounded by planes.

Prop. 12.

Let pumv (fig. 5) represent the same pyramid as in *Prop.* 7; to find its attraction on a point p′ in the produced axis.

Suppose p′u, p′v joined, the attraction of the pyramid pumv, on the point p′, is the difference of the attractions of the pyramids p′umv, p′upv on the same point; which point being at the common vertex of these two pyramids, their attractions are found by Propositions 7 and 9; and the problem is solved.

Prop. 13.

Let it now be the action of the pyramid pumv, fig. 7, (where the plane of the base umv is not perpendicular to the line mpp′) on the point p′, that is required.

The attraction sought for will still be the difference of the actions of the two pyramids p′umv, p′upv, but these must now *both* be found by *Prop.* 9.

Prop. 14.

Figure 6 representing, in every respect, the same as in *Prop.* 8, it is required to find the action of the pyramid, on the point p′, in the produced axis.

As this solid may be divided into others, situated, with respect to p′, like that in *Prop.* 12, the problem is solved by what was shewn there.

If the attracted point was not in the line p′pm, perpendicular to the base, but in some other line πpμ, passing through the vertex, and meeting the base in μ; draw lines from μ to all the angles of the base, and the solid will be divided in such pyramids as were treated of in *Prop.* 13.

Prop. 15.

Let pumv, fig. 7, be any triangular pyramid whatever, and let it be any how cut by a plane, whose intersection with the pyramid is the triangle αβγ; it is required to determine the action of the portion φβγvum (which is cut off by the plane) on a point at p.

The attraction, of the solid in question, is the difference of the actions of the pyramids pumv and pαβγ, which actions are found by *Prop.* 9.

Prop. 16, *or general Problem.*

To find the action of any solid, bounded by planes, on a point either within or without it.

It is plain, that by drawing lines, from the attracted point through the solid, this may always be divided, either into such pyramids (with the point at the vertex) as were considered in

Prop. 9, or into portions of pyramids, like that treated of in *Prop.* 15; and consequently the solution of the problem may be obtained by means of these propositions.

§. III.

Of the Attraction of Prisms.

Prop. 17.

To find the attraction of a right prism, whose base is a regular polygon, on a point in the produced axis.

We saw, in *Prop.* 5, that the action of a regular polygon, on a point situated perpendicularly over its centre is

$$A = 2n \text{ arc } \left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{1+r^2}{a^2}b^2}\right) - (n-2)\pi.$$

To find the attraction of the prism, change a into x, multiply by $\dot{x}$, and take the fluent.

Now $\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)\,b^2}\right) = \int \dot{x}\,\varphi\,(x, b, r)$ $= x \text{ arc } \left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)\,b^2}\right) - \int x\,\dot{\overline{\text{arc}}}\,\left(\text{tang.} = \frac{1}{rx}\right.$ $\left.\sqrt{x^2 + (1+r^2)\,b^2}\right) = x \text{ arc } \left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)\,b^2}\right)$ $+ \int \frac{rb^2 x\dot{x}}{(b^2+x^2)\sqrt{(1+r^2)\,b^2+x^2}}$, because $\dot{\overline{\text{arc}}} = \frac{\dot{\overline{\text{tang.}}}}{1+\text{tang.}^2}$; and, by taking the fluent,* it becomes $= x \text{ arc } \left(\text{tang.} = \frac{1}{rx}\right.$ $\left.\sqrt{x^2 + (1+r^2)\,b^2}\right) - b\text{L}.\,\frac{rb + \sqrt{(1+r^2)\,b^2+x^2}}{\sqrt{b^2+x^2}}$.

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* Put $x^2 + (1+r^2)\,b^2 = z^2$, $b^2 + x^2 = z^2 - r^2b^2$, $x\dot{x} - z\dot{z}$; then $\int \frac{rb^2 x\dot{x}}{(b^2+x^2)\sqrt{(1+r^2)\,b^2+x^2}} = \int \frac{rb^2\dot{z}}{z^2 - r^2b^2} = \frac{b}{2}\text{L}.\,\frac{z+rb}{z-rb}$ (SIMPSON's Fluxions, p. 140,) $= \frac{b}{2}\text{L}.\,\frac{(z+rb)^2}{z^2-r^2b^2} = b\text{L}.\,\frac{\sqrt{(1+r^2)\,b^2+x^2} + rb}{\sqrt{b^2+x^2}}$.

For the sake of brevity, call this quantity F (b, r, x) and we have for the attraction of the prism,

$$A = 2n\mathrm{F}\,(b, r, x) - (n - 2)\,\pi x + \text{corr.}$$

The attraction of any other right prism, *in the direction of its length*, depends on the same function F (b, r, x); as in

Prop. 18.

To find the attraction of a right prism, whose base is a rectangle, on a point in the produced axis.

We saw in *Prop.* 4, that the action of a rectangle, on a point situated perpendicularly over its centre is

$$A = 4\varphi\,(a, b, r) + 4\varphi\,(a, b', r') - 2\pi,$$

where b and b' are the halves of the sides of the rectangle, and r and r' the tangents of the angles, formed respectively by those sides and the diagonal. By changing a into x, and multiplying by $\dot{x}$, we have, for the prism,

$$A = 4\int \dot{x}\varphi\,(x, b, r) + 4\int \dot{x}\varphi\,(x, b', r') - 2\pi x;$$ whence, by what was done in the last proposition,

$$A = 4\mathrm{F}\,(b, r, x) + 4\mathrm{F}\,(b', r', x) - 2\pi x + \text{corr.}$$

Prop. 19.

Let the base of the prism be a rhombus, the attracted point in the produced axis of the prism.

We found, in *Cor.* 1, *Prop.* 2, that the action of a rhombus on a point situated perpendicularly over its centre, is (keeping the notation there used)

$$A = 4\varphi\,(a, b, r) + 4\varphi\,(a, b, r') - 2\pi,$$ therefore, proceeding as before, we have, for the prism,

$$A = 4\mathrm{F}\,(b, r, x) + 4\mathrm{F}\,(b, r', x) - 2\pi x + \text{corr.}$$

We will now consider the attractions of prisms more generally.

Prop. 20.

Let pr′v′vrm, fig. 8, be a right prism, whose base is the triangle vrm right angled at r. It is required to determine its attraction on a point at p, in the directions pm, pr′, po; po being parallel to r′v′.

If we wish to obtain a solution by means of what has been already done, we may conceive the solid under consideration to be divided into two pyramids; viz. the pyramid pmrv with the triangular base mrv; and the other pyramid pr′v′vr, whose base is the rectangle r′v′vr; the point p being at the common vertex of both.

Put pm $= x$, mr $=$ pr′ $= x'$; conceive the diagonal vr′ to be drawn, and put tang. rmv $=$ tang. r′pv′ $= r$; tang. v′rr $= r'$; tang. vr′v′ $= r''$; tang. rpm $= m$; tang. rpr′ $= m'$.

We get immediately, from propositions 7 and 10 (putting A, B, C for the respective actions in the directions pm, pr′, po)

$$A = x\varphi\,(1, m, r) - x \text{ arc } \left(\text{tang.} = \frac{1}{r}\right) + x'\chi\,(1, m', r') + x'\psi\,(1, r, r'');$$

$$B = x\chi\,(1, m, r) + x'\varphi\,(1, m', r') + x'\varphi\,(1, r, r'') - x'\frac{\pi}{2};$$

$$C = x\psi\,(1, m, r) + x'\psi\,(1, m', r') + x'\chi\,)1, r, r'').$$

In finding these values, the first expressions of *Prop.* 10 were made use of; if we take the others, there will result the simpler forms

$$A = x\varphi\,(1, m, r) - x \text{ arc } \left(\text{tang.} = \frac{1}{r}\right) + x'\left\{L.\left(\sqrt{1+r^2}+r\right) - L.\frac{\sqrt{1+r^2+m'^2}+r}{\sqrt{1+m'^2}}\right\};$$

$$B = x\chi\,(1, m, r) + x' \text{ arc } \left(\text{tang.} = \frac{rm'}{\sqrt{1+r^2+m'^2}}\right);$$

$$C = x\psi\,(1, m, r) + x'\left\{L\,.\,(\sqrt{1+m'^2} + m') - L\,.\,\frac{\sqrt{1+m'^2+r^2} + m'}{\sqrt{1+r^2}}\right\}.$$

These are, perhaps, as simple expressions as can be had in this case; we may however find others; nor is it necessary to conceive the solid divided into pyramids.

Thus we may obtain the value of the force A, by the method of the preceding propositions: a triangular section of the prism, parallel to the end r′pv′, and at a distance x from that end, will have its action on p expressed by $\varphi\,(x, x', r) - \text{arc}\left(\text{tang.} = \frac{1}{r}\right)$, using the same notation as before; therefore, the action of the pyramid, in the direction pm, is

$$A = \int \dot{x}\varphi\,(x, x', r) - x\,\text{arc}\left(\text{tang.} = \frac{1}{r}\right) = F\,(x', r, x) - x\,\text{arc}\left(\text{tang.} = \frac{1}{r}\right) + \text{corr.}$$

Another Way of finding the force B.

Draw ks parallel to r′v′; call pk, x'; then ks $= r \times x'$. Conceive a plane to pass through ks, parallel to the back rvv′r′ of the prism; the section made by it will be a rectangle whose sides are x and rx'. The action of this rectangle on p, in the direction pr′, is $\text{arc}\left(\text{tang.} = \frac{rx}{\sqrt{x^2 + r^2x'^2 + x'^2}}\right)$ by *Prop.* 4, wherefore the action of the prism, in the same direction, is

$B = \int \dot{x}'\,\text{arc}\left(\text{tang.} = \frac{rx}{\sqrt{x^2 + (1+r^2)\,x'^2}}\right)$ where x' is the variable quantity, we have then

$$B = x'\,\text{arc}\left(\text{tang.} = \frac{rx}{\sqrt{x^2 + (1+r^2)\,x'^2}}\right) - \int x'\,\dot{\overline{\text{arc}}}\left(\text{tang.} = \frac{rx}{\sqrt{x^2 + (1+r^2)\,x'^2}}\right).$$

Put $\frac{x^2}{1+r^2} = c^2$, and the last term becomes $-\int x' \, \dot{\overline{\text{arc}}} \, \Big(\text{tang.}$

$$\left.\frac{rc}{\sqrt{c^2+x'^2}}\right) = + rc \int \left\{ \frac{\dot{x}'}{\sqrt{c^2+x'^2}} - \frac{(1+r^2)\,c^2\dot{x}'}{\left\{(1+r^2)\,c^2+x'^2\right\}\sqrt{c^2+x'^2}} \right\} = rc$$

$$\left\{ \mathrm{L}\left(x' + \sqrt{c^2+x'^2}\right) - \frac{\sqrt{1+r^2}}{r}\,\mathrm{L}\left(\frac{x'}{\sqrt{x^2+x'^2}} + \sqrt{\frac{1}{r^2} + \frac{x'^2}{x^2+x'^2}}\right) \right\};$$

therefore

$$\mathrm{B} = x' \text{ arc }\left(\text{tang.} = \frac{rx}{\sqrt{x^2+(1+r^2)\,x'^2}}\right) + rc\,\mathrm{L}\left(x' + \sqrt{c^2+x'^2}\right)$$

$$- x\,\mathrm{L}\left(\frac{x'}{\sqrt{x^2+x'^2}} + \sqrt{\frac{1}{r^2} + \frac{x'^2}{x^2+x'^2}}\right) + \text{corr.}$$

If the fluent is to begin when $x' = o$, the correction is

$$- rc\,\mathrm{L}\,.\,c + x\,\mathrm{L}\,\frac{1}{r}.$$

I have dwelt the longer on this proposition, because the attraction of right prisms, in all cases, may be made to depend on it.

Cor. 1. It is in the first place evident, that by means of this proposition, we may find, by parts, the force, both in quantity and direction, with which a point q, any where on the edge pm of the scalene prism represented by fig. 9, is attracted.

The same may be said of the action on p′ any where in the produced axis; this will be the difference of the actions of two prisms, like that in the figure.

Cor. 2. Moreover, if, instead of the prism in fig. 9, the point q was placed any where on the edge of a prism whose base is the triangle quv, fig. 10, the action may still be found; for it will now depend on the difference of the action of such prisms as were treated of in the Proposition; that is to say, the action of the prism whose base is quv as the difference of the actions of those whose bases are qvr, qur, qr being a perpendicular on vu produced.

Prop. 21.

To find the quantity and direction of the attraction of a right prism, whose base is any triangle whatever, on a point any where on its surface.

Let the triangle uvu′, fig. 11, be a section of the prism, parallel to its base, and through the attracted point p. Let fall the perpendiculars pr, pr′, on the opposite sides: and the solid may be divided into four prisms, whose bases are the right angled triangles pur, prv, pvr′, pr′u′, and the attraction of each of these, both in quantity and direction, is given by *Prop.* 20.

It is plain that there may be other cases of this problem, besides the one here considered; for instance, one of the perpendiculars may fall beyond the base; but it would be endless, in a subject of this kind, to consider every particular case, and in none can the intelligent reader find the smallest difficulty.

Prop. 22.

To find the attraction of any prism, fig. 12, whose base is a *convex* polygon uvu′v′, on a point q any where within it.

As such a solid may be divided into triangular prisms, like those in *Prop.* 20 and its corollaries, with the attracted point on the common edge pm, the problem is already solved.

If the point be at p′, in the line mp produced, the action on it may still be found, being the difference of the actions of two prisms, like that in the figure.

Prop. 23.

Let vuv′, fig. 13, be the section of an isosceles prism; pv a line passing through the vertex v perpendicularly on the middle of the base. It is required to find the action of the prism on any point p in the line pv.

This action is equal to the action of the prism whose base is the triangle v′pu (given by *Prop.* 20), less the actions of the prisms whose bases, or sections, are the triangles v′vp, uvp, which are found by *Cor.* 2, *Prop.* 20.

Scholium.

I considered here only an isosceles prism; but the solution is just the same, if the section of the prism is any triangle whatever, as v′uv, fig. 14, and the action on a point p, (situated in the line uv produced) is required. For the attraction wanted will be the difference of attractions of the two prisms whose bases are the triangles v′up, v′vp, and these are given by *Cor.* 2, *Prop.* 20.

Suppose the base of the prism, whose attraction is required, to be the trapezium v′uαβ, fig. 14, the action of this on p, being the difference of the actions of the triangular prisms, whose bases are v′uv, βαv, is found by the case just now considered.

In this manner, might cases be multiplied without end; but I think it is sufficiently plain, that by means of the preceding propositions and scholium, we may find the action of any prism whatever, on a point either within or without it.

§. IV.

Of the Attraction of certain Solids not terminated by Planes.

The expressions, arrived at in the first section, are useful in finding the attraction, not only of such solids as are bounded by planes, but of a great variety of others; viz. of such as have their sections in one direction continuous curves, whilst, being cut in a different way, there results, from their intersection with a plane, a polygon, or rectangle, or some other right lined figure.

As no one, that I know of, has considered the actions of such bodies, I shall offer no apology for giving a few examples.

Let uvv′u′, fig. 15, represent any regular polygon, whose plane is perpendicular to the line pm, and its centre in that line: moreover, let this polygon be variable in magnitude, and move parallel to itself in the direction pm, in such a manner, that the middle point r of each of its sides uv, may describe a given curve pr.

Prop. 24.

Let it be required to find the attraction of the solid thus generated by the polygon, when the curve pr is a circle,* and the attracted point at the vertex p of the solid.

The attraction of a regular polygon was found in *Prop.* 5 and it will be adapted to our present purpose, by putting x^2 for a^2, and $2kx - x^2$ for b^2, where k is the radius of the circle pr: and we have, for the attraction of the solid,

$$A = 2n\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{1+r^2}{x^2}(2kx - x^2)}\right) - (n-2)\pi x$$

$$\text{or } A = 2n\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{1}{rx}\sqrt{2k(1+r^2)x - r^2x^2}\right) - (n-2)\pi x.$$

That part of A, under the sign of integration, equals

$$2nx \text{ arc } \left(\text{tang.} = \frac{1}{rx}\sqrt{2k(1+r^2)x - r^2x^2}\right) - 2n\int x \, \overset{\cdot}{\overline{\text{arc}}} \left(\text{tang.} = \frac{1}{rx}\sqrt{2k(1+r^2)x - r^2x^2}\right),$$

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* We may, not improperly, term this solid *a polygonal sphere.*

and, if we put $zz = x$, the last term becomes

$$- 2n\int z^2\, \dot{\overline{\text{arc}}}\left(\text{tang.} = \frac{1}{rz}\sqrt{2k(1+r^2) - r^2z^2}\right)$$

$$= 2n\int \frac{rz^2\dot{z}}{\sqrt{2k(1+r^2) - r^2z^2}} = 2n\int \frac{z^2\dot{z}}{\sqrt{\frac{2k(1+r^2)}{r^2} - z^2}};$$ this fluent is

$$2n\left\{\frac{k(1+r^2)}{r^2}\text{ arc }\left(\text{sine} = \frac{rz}{\sqrt{2k(1+r^2)}}\right) - \frac{1}{2}z\sqrt{\frac{2k(1+r^2)}{r^2} - z^2}\right\}$$

or by putting its value for z,

$$\frac{2nk(1+r^2)}{r^2}\text{ arc }\left(\text{sine} = \frac{r\sqrt{x}}{\sqrt{2k(1+r^2)}}\right) - n\sqrt{\frac{2k(1+r^2)}{r^2}x - x^2}.$$

Collecting all the parts of A, we have at length

$$A = 2nx \text{ arc }\left(\text{tang.} = \frac{1}{rx}\sqrt{2k(1+r^2)x - r^2x^2}\right) - \frac{n}{r}\sqrt{2k(1+r^2)x - r^2x^2} + \frac{2nk(1+r^2)}{r^2}\text{ arc }\left(\text{sine} = \frac{r\sqrt{x}}{\sqrt{2k(1+r^2)}}\right) - (n-2)\pi x + \text{corr.}$$

But it is easy to see that each of the arcs in this expression is the complement of the other; put then $A = \text{arc}\left(\text{tang.} = \frac{1}{rx}\sqrt{2k(1+r^2)x - r^2x^2}\right)$ and the expression becomes

$$A = \left\{\frac{nk}{r^2} + n(k-x)\right\}(\pi - 2A) + 2x\pi - nx\,.\,\text{tang.}\,A + \text{corr.}$$

When $x = o$, $A = \frac{\pi}{2}$, so that, if the fluent is to begin when $x = o$, no correction is necessary.

When $x = 2k$, $A = \text{arc}\left(\text{tang.} = \frac{1}{r}\right) = \frac{(n-2)\pi}{2n}$, and $\sqrt{2k(1+r^2)x - r^2x^2} = 2k$, $\pi - 2A = \pi - \frac{(n-2)\pi}{n} = \frac{2\pi}{n}$: whence we have for *the attraction of the whole solid*

$$A = \frac{1+r^2}{r^2} \times 2k\pi - \frac{2nk}{r} \ldots\ldots\ldots\ldots (\alpha).$$

This will appear to be an expression of great simplicity, if we reflect what very different solids it belongs to, from that whose section is a triangle, to the sphere whose section is a circle.

In the latter case n is infinitely great and r infinitely small; r is also the tangent of $\frac{\pi}{n} = \frac{\pi}{n} + \frac{1}{3} \cdot \frac{\pi^3}{n^3}$, if we reject the other terms of the expansion, on account of their smallness.

The expression for A may then, in this case, be put into the form

$$A = 2k\pi + \frac{2k\pi}{r\left\{\frac{\pi}{n} + \frac{1}{3} \cdot \frac{\pi^3}{n^3}\right\}} - \frac{2nk}{r};$$

now $r\left\{\frac{\pi}{n} + \frac{1}{3} \cdot \frac{\pi^3}{n^3}\right\} = (q \cdot p)\, r\left\{\frac{\pi}{n} + \frac{1}{3} \cdot \frac{\pi}{n} r^2\right\} = r\left(1 + \frac{r^2}{3}\right)\frac{\pi}{n}$, whence the second term of the last number is changed into

$$\frac{2kn}{r\left(1 + \frac{r^2}{3}\right)} = \frac{2kn\left(1 - \frac{r^2}{3}\right)}{r} = \frac{2kn}{r} - \frac{2}{3}knr = \frac{2nk}{r} - \frac{2}{3}k\pi, \text{ by}$$

putting π for nr: and, by substituting this value, we have at last $A = \frac{4}{3} k\pi$; which is the well known expression for the attraction of a sphere on a point at its surface.

If the generating polygon is a square instead of a circle, $r = \text{tang. } 45^\circ = 1$, and equation ($\alpha$) gives $A = 4k(\pi - 2) = 4k \times 1{,}14159$, &c. which exceeds the attraction of the sphere by about one-tenth, if pr is the same circle in both.

Cor. If we would know the radius (k') of a sphere, which shall attract, a point at its surface, as much as a polygonal sphere, of the length $2k$, does a point at its vertex, we have only to put

$$\frac{4}{3}\pi k' = \frac{1+r^2}{r^2}\, 2\pi k - \frac{2nk}{r}, \text{ whence}$$

$$k' = \frac{3}{2}k\left\{\frac{1+r^2}{r^2} - \frac{n}{\pi r}\right\}.$$

Prop. 25.

Let the directing curve pr be a parabola; the attracted point at the vertex p of the solid.

We must here make use of the formula in *Prop.* 5, as we did in the last example; but as the equation of a parabola is $y^2 = \alpha x$, this latter quantity must be put for b^2. Thus we get, for the attraction of the solid,

$$A = 2n\int\dot{x} \text{ arc } \left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1 + r^2)\alpha x}\right) - (n-2)\pi x.$$

The part, having the sign of integration, may be put under the form $2nx$ arc. $\left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1 + r^2)\alpha x}\right)$ $2n\int x \dot{\overline{\text{arc}}}$ $\left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1 + r^2)\alpha x}\right)$; in the last term of which put $z^2 = x$, and it will become

$$-2n\int z^2 \dot{\overline{\text{arc}}} \left(\text{tang.} = \frac{1}{rz}\sqrt{z^2 + (1 + r^2)\alpha}\right)$$

$$= 2n\alpha r\int\left\{\frac{\dot{z}}{\sqrt{z^2 + (1+r^2)\alpha}} - \frac{\alpha\dot{z}}{(z^2+\alpha)\sqrt{z^2 + (1+r^2)\alpha}}\right\}$$

$$= 2n\alpha r\left\{\text{L}\left(z + \sqrt{z^2 + (1 + r^2)\alpha}\right) - \frac{1}{r}\text{ arc }\left(\text{sine} = \frac{r}{\sqrt{1+r^2}} \times \frac{z}{\sqrt{z^2+\alpha}}\right)\right\}.$$

Collecting all the terms, we have at length

$$A = 2nx \text{ arc } \left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)\alpha x}\right) - 2n\alpha \text{ arc }\left(\text{sine} = \frac{r}{\sqrt{1+r^2}} \times \frac{\sqrt{x}}{\sqrt{x+\alpha}}\right) + 2n\alpha r \text{ L }\left(\sqrt{x} + \sqrt{x + (1 + r^2)\alpha}\right) - (n - 2)\pi x + \text{corr.}$$

It is observable here, as in the last proposition, that each of the arcs in this expression is the complement of the other; put $A = \text{arc }\left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)\alpha x}\right)$, and the attraction becomes $A = n\,(x+\alpha)\,2A-\pi) + 2x\pi + 2n\alpha r\,L\,(\sqrt{x} + \sqrt{x+(1+r^2)\alpha}) +$ corr.

When $x = 0$, $A = \frac{\pi}{2}$; so that, if the fluent is to begin at that term, we have corr. $= -2n\alpha r\,L\,\sqrt{(1+r^2)\,\alpha}$.

If we would find, from this expression, the attraction of the limit of these solids (which is the parabolic conoid) we must observe, that the arc A may be put under the form $\frac{\pi}{2} -$ arc (sine $= \frac{r}{\sqrt{1+r^2}} \times \frac{\sqrt{x}}{\sqrt{x+\alpha}}$), whence, because r is infinitely small, $2A - \pi = -2r\,.\,\frac{\sqrt{x}}{\sqrt{x+\alpha}} = -2\,\frac{\pi}{n}\,.\,\frac{\sqrt{x}}{\sqrt{x+\alpha}}$, qu. prox.; substituting this value, and π for nr, and neglecting r^2, we get

$$A = 2\pi\left\{x - \sqrt{x^2+\alpha x} + \alpha L\,(\sqrt{x} + \sqrt{x+\alpha})\right\} + \text{corr.}$$

for the action of a parabolic conoid on a point at its vertex.

Prop. 26.

Let the curve pr be a parabola *convex* to the axis pm, in which case $y = \frac{x^2}{\alpha}$; and we have, by proceeding as before,

$A = 2n\int\dot{x}$ arc $(\text{tang.} = \frac{1}{r}\sqrt{1+\frac{(1+r^2)}{\alpha^2}x^2}) - (n-2)\,\pi x$; or

$A = 2nx$ arc $(\text{tang.} = \frac{1}{r}\sqrt{1+\frac{1+r^2}{\alpha^2}x^2})\;2n\int x\,\overset{\cdot}{\overline{\text{arc}}}\,(\text{tang.} = \frac{1}{r}\sqrt{1+\frac{1+r^2}{\alpha^2}x^2}) - (n-2)\,\pi x$; if we put $a^2 = \frac{\alpha^2}{1+r^2}$, the term, under the integral sign, becomes

$$-\,2n\int x\,\overset{\cdot}{\overline{\text{arc}}}\,(\text{tang.} = \frac{1}{ar}\sqrt{a^2+x^2})$$

$$= -\,2nar\int\left[\frac{\dot{x}}{\sqrt{a^2+x^2}} - \frac{(1+r^2)\,a^2\dot{x}}{\left\{(1+r^2)\,a^2+x^2\right\}\sqrt{a^2+x^2}}\right].$$

$$= -2nar\left\{\mathrm{L}\left(x+\sqrt{a^2+x^2}\right)-\frac{\sqrt{1+r^2}}{r}\,\mathrm{L}\left(\frac{x}{\sqrt{x^2+a^2}}+\sqrt{\frac{1}{r^2}+\frac{x^2}{x^2+a^2}}\right)\right\}$$; so that

$$\mathrm{A}=2nx\ \text{arc}\left(\text{tang.}=\frac{1}{r}\cdot\sqrt{1+\frac{1+r^2}{a^2}x^2}\right)-(n-2)\pi x+\text{corr.}$$

$$-2nr\frac{a}{\sqrt{1+r^2}}\,\mathrm{L}\left(x+\sqrt{\frac{a^2}{1+r^2}+x^2}\right)+2n\alpha\mathrm{L}\left\{\frac{x}{\sqrt{x^2+a^2}}+\sqrt{\frac{1}{r^2}+\frac{x^2}{x^2+a^2}}\right\}.$$

This is the attraction of a polygonal parabolic spike, on a particle at the point. When the polygon becomes a circle,

$$\text{Arc}\left(\text{tang.}=\frac{1}{r}\sqrt{1+\frac{1+r^2}{a^2}x^2}\right)=\frac{\pi}{2}-\frac{r\alpha}{\sqrt{\alpha^2+x^2}},$$ and

$$\mathrm{L}\left\{\frac{x}{\sqrt{x^2+\alpha^2}}+\sqrt{\frac{1}{r^2}+\frac{x^2}{x^2+\alpha^2}}\right\}=\mathrm{L}\frac{1}{r}+\frac{rx}{\sqrt{\alpha^2+x^2}},$$ whence it will easily appear that, in the case under consideration,

$$\mathrm{A}=2\pi\left\{x-\alpha\mathrm{L}\left(x+\sqrt{\alpha^2+x^2}\right)\right\}+\text{corr.}$$

We may conceive the plane uvv′u′, fig. 15, instead of a regular polygon to be a rectangle, moving along the line pm, as in the former case, with its centre in that line; and, with the middle points r and r′ of its sides, touching curves pr, pr′ either of the same or different kinds.

The section of the generated solid, or *groin*, perpendicular to its axis, will have its action on the point p (if we put $x =$ pm, and b and b' for the sides of the rectangle) expressed by

$$4\ \text{arc}\left(\text{tang.}=\frac{bb'}{x\sqrt{x^2+b^2+b'^2}}\right);$$

and if we multiply this by $\dot{x}$, and put for b and b' their values, given by the equations of the curves pr, pr′, the fluent will be the attraction of the generated solid.

Prop. 27.

Let pr be a circle, pr′ a parabola; to find the attraction of the solid on a point p at its vertex.

Let the radius of the circle be k, the parameter of the parabola α: then we have $b = \sqrt{2kx - x^2}$; $b' = \sqrt{\alpha x}$, and

$$A = 4\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{\sqrt{\alpha x}\sqrt{2kx - x^2}}{x\sqrt{(2k+\alpha)x}}\right),$$

or $A = 4\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{c\sqrt{2k-x}}{\sqrt{x}}\right)$, if we put $c = \frac{\sqrt{\alpha}}{\sqrt{2k+\alpha}}$, and taking the fluent by parts

$$A = 4x \text{ arc } \left(\text{tang.} = \frac{c\sqrt{2k-x}}{\sqrt{x}}\right) - 4\int x\, \overset{\cdot}{\overline{\text{arc}}} \left(\text{tang.} = \frac{c\sqrt{2k-x}}{\sqrt{x}}\right),$$

the last term of which, if we put z^2 for x, becomes

$$-4\int z^2\, \overset{\cdot}{\overline{\text{arc}}} \left(\text{tang.} = \frac{c\sqrt{2k-z^2}}{z}\right)$$ and this, by actually taking the fluxion of the arc $= \int \frac{8kcz^2\dot{z}}{\left\{(1+c^2)z^2 + 2kc^2\right\}\sqrt{2k-z^2}}$, or, by restoring the value of c, $= 4\sqrt{\alpha}\,.\,\sqrt{2k+\alpha}\,.\int \frac{z^2\dot{z}}{(z^2+\alpha)\sqrt{2k-z^2}}$, or by division

$$= 4\sqrt{\alpha}\,.\,\sqrt{2k+\alpha}\,.\int\left\{\frac{\dot{z}}{\sqrt{2k-z^2}} - \frac{\alpha\dot{z}}{(z^2+\alpha)\sqrt{2k-z^2}}\right\}$$

$= 4\sqrt{\alpha}\sqrt{2k+\alpha}$ arc $\left(\text{sine} = \frac{z}{\sqrt{2k}}\right) - 4\alpha$ arc $\left(\text{sine} = \frac{\sqrt{2k+\alpha}}{\sqrt{2k}} \times \frac{z}{\sqrt{z^2+\alpha}}\right)$;

so that we have at last

$$A = 4x \text{ arc } \left(\text{tang.} = \frac{\sqrt{\alpha}}{\sqrt{2k+\alpha}} \times \frac{\sqrt{2k-x}}{\sqrt{x}}\right) + 4\sqrt{\alpha}\sqrt{2k+\alpha} \text{ arc } \left(\text{sine} = \frac{\sqrt{x}}{\sqrt{2k}}\right) - 4\alpha \text{ arc } \left(\text{sine} = \frac{\sqrt{2k+\alpha}}{\sqrt{2k}} \times \frac{\sqrt{x}}{\sqrt{x+\alpha}}\right) + \text{corr.}\ldots\ldots(\beta).$$

As this expression vanishes when $x = o$, if the fluent is to begin at that value of x, no correction is to be added. The last of the arcs, in the expression above, is the complement of the first; denote then the first by A, and we have

$$A = 4(x+\alpha)A - 2\alpha\pi + 4\sqrt{\alpha}\sqrt{2k+\alpha}\ \text{arc}\left(\text{sine} = \frac{\sqrt{x}}{\sqrt{2k}}\right);$$

lastly, if we want the whole fluent, when $x = 2k$, we get

$$A = 2\pi\left(\sqrt{2k\alpha+\alpha^2} - \alpha\right).$$

Cor. 1. If we make α infinite, in this last value of A, it becomes $A = 2k\pi$; which is the action of *an infinitely long circular cylinder*, on a point at its surface. This is the attraction of the whole cylinder when $x = 2k$; to find the same for *any* value of x, make α infinite in formula (β); this gives

$$A = 4x\ \text{arc}\left(\text{tang.} = \frac{\sqrt{2k-x}}{\sqrt{x}}\right) + 4(\alpha+k)\ \text{arc}\left(\text{sine} = \frac{\sqrt{x}}{\sqrt{2k}}\right)$$
$$-4\alpha\ \text{arc}\left\{\text{sine} = \left(1+\frac{2k-x}{2\alpha}\right)\frac{\sqrt{x}}{\sqrt{2k}}\right\};$$ but (Euleri Calc. Diff. p. 376)

$$\text{Arc}\left\{\text{sine} = \left(1+\frac{2k-x}{2\alpha}\right)\frac{\sqrt{x}}{\sqrt{2k}}\right\} = \text{arc}\left(\text{sine} = \frac{\sqrt{x}}{\sqrt{2k}}\right) + \frac{\sqrt{2kx-x^2}}{2\alpha}$$

qu. prox.; the substitution of this value gives

$$A = 4x\ \text{arc}\left(\text{tang.} = \frac{\sqrt{2k-x}}{\sqrt{x}} + 4k\ \text{arc}\left(\text{sine} = \frac{\sqrt{x}}{\sqrt{2k}}\right) - 2\sqrt{2kx-x^2},\right.$$

which, because the latter arc is the complement of the former, is changed to

$$A = 2k\pi - 4(k-x)\ \text{arc}\left(\text{tang.} = \frac{\sqrt{2k-x}}{\sqrt{x}}\right) - 2\sqrt{2kx-x^2}.$$

Cor. 2. In like manner we may find the *attraction of an infinitely long parabolic cylinder*, on a point in its surface, at the vertex of the parabola; this is effected by making k infinite in formula (β), whence there results

$$A = 4x\ \text{arc}\left(\text{tang.} = \frac{\sqrt{\alpha}}{\sqrt{x}}\right) - 4\alpha\ \text{arc}\left(\text{tang.} = \frac{\sqrt{x}}{\sqrt{\alpha}}\right) + 4\sqrt{\alpha x};$$ or

$$A = 4\,(x + \alpha)\text{ arc }\left(\text{tang.} = \frac{\sqrt{\alpha}}{\sqrt{x}}\right) + 4\sqrt{\alpha x} - 2\alpha\pi.$$

Cor. 3. If $\alpha = \frac{2}{3}k$, the attraction of the solid in the proposition equals that of a sphere whose radius is k; for by substituting $\frac{2}{3}k$ for α, in the expression $2\pi\,(\sqrt{2k\alpha + \alpha^2} - \alpha)$, it becomes $\frac{4}{3}k\pi$, the action of a sphere whose radius is k on a point in its surface.

The attractions of cylinders of finite length, in directions perpendicular to their axes, are to be found after the manner of this last proposition; but there are not many cases in which they can be expressed by circular arcs and logarithms.

Prop. 28.

Let fig. 16 represent a circle, C the centre, ab, cd two parallel chords; conceive a right cylinder, whose section is the portion abcd of the circle, terminated by the chords ab, cd, to be extended to the distance d above and below the plane of the figure.

It is required to determine the action of this cylinder on a point at C.

Put k for the radius of the circle, and let x be the distance from C of a chord parallel to ab. Then, using the same formula as in the last *Prop.* we have $b = d$, $b' = \sqrt{k^2 - x^2}$, and for the action of the solid, $A = 4\int\dot{x}\text{ arc }\left(\text{tang.} = \frac{d\sqrt{k^2 - x^2}}{x\sqrt{d^2 + k^2}}\right)$ or

$$A = 4x\text{ arc }\left(\text{tang.} = \frac{d\sqrt{k^2 - x^2}}{x\sqrt{d^2 + k^2}}\right) - 4\int x\,\dot{\overline{\text{arc}}}\left(\text{tang.} = \frac{d\sqrt{k^2 - x^2}}{x\sqrt{d^2 + k^2}}\right)$$

$$= 4x\text{ arc }\left(\text{tang.} = \frac{d\sqrt{k^2 - x^2}}{x\sqrt{d^2 + k^2}}\right) - 4d\text{L}\,.\,\frac{\sqrt{d^2 + k^2} + \sqrt{k^2 - x^2}}{\sqrt{d^2 + x^2}} + \text{corr.}$$

If the fluent is to begin when $x = o$, the correction is

$4dL.\frac{\sqrt{d^2+k^2}+k}{d}$; and the fluent, taken from $x=o$ to $x=k$, is

$$A = 4dL\frac{\sqrt{d^2+k^2}+k}{d},$$

which agrees with what Mr. PLAYFAIR found in a different manner; except in this case, the object of the present proposition is different from that of his, which finds the action of such portions of the cylinder as have *sectors* for their bases.

Prop. 29.

Let the base of the cylinder be the figure parmnbp, fig. 17, the curves par, pbn being inverted parabolas; or in which $pm^2 = \alpha \times rm$. Let the attracted point be at p; and let the cylinder be extended to the distance d above and below the plane of the figure.

Using the same formula as before, and putting $pm = x$, we have $b = d$, $b' = \frac{x^2}{\alpha}$; and, for the action of the solid,

$$A = 4\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{dx}{\sqrt{\alpha^2 d^2 + \alpha^2 x^2 + x^4}}\right), \text{ or}$$

$$A = 4x \text{ arc } \left(\text{tang.} = \frac{dx}{\sqrt{\alpha^2 d^2 + \alpha^2 x^2 + x^4}}\right) - 4\int x \; \dot{\overline{\text{arc}}} \left(\text{tang.} = \frac{dx}{\sqrt{\alpha^2 d^2 + \alpha^2 x^2 + x^4}}\right);$$

the last term of which becomes, by taking the fluxion of the arc,

$$-4\int \frac{\alpha^2 d^3 x\dot{x} - dx^5 \dot{x}}{\left\{\alpha^2 d^2 + (\alpha^2 + d^2)x^2 + x^4\right\} \sqrt{\alpha^2 d^2 + \alpha^2 x^2 + x^4}}$$

an expression integrable by circular arcs and logarithms.

When d is infinite, this fluent takes a very simple form, viz.

$$-4\int \frac{\alpha x \dot{x}}{\alpha^2 + x^2} = -2\alpha \int \frac{\frac{2x\dot{x}}{\alpha^2}}{1 + \frac{x^2}{\alpha^2}} = -2\alpha L\left(1 + \frac{x^2}{\alpha^2}\right), \text{ and, in}$$

this case

$$A = 4x \text{ arc } \left(\text{tang.} = \frac{x}{\alpha}\right) - 2\alpha L \left(1 + \frac{x^2}{\alpha^2}\right).$$

Cor. 1. Draw the right lines pr, pn; because $\frac{x}{\alpha} = \frac{\text{pm}}{\alpha} = \frac{\text{rm}}{\text{pm}}$ = tang, rpm, the last expression may be put into the form

$$A = 4x \text{ arc. rpm} - 4\alpha L \text{. sec. rpm.}$$

Cor. 2. If, in the last value of B in *Prop.* 20, we make x infinite, there results $B = x' \text{ arc } (\text{tang.} = r)$; from whence it is plain that the first term of the expression for A in the last cor. viz. $4x$ arc rpm, expresses the action of an infinitely long prism, whose base is the triangle rpn, on the point p.

Consequently, the other term of A, or $4\alpha L$. sect. rpm, is the action of the infinitely long solid whose base consists of the parabolic segments parp, pbnp.

We may next consider the generating plane uvv′u′, fig. 15, to be a rhombus, given in *species*, and so varying in magnitude, as to touch four similar and equal curves, at those points where perpendiculars from the centre of the rhombus fall on its sides.

Prop. 30.

Let the guiding curves be semi-circles, to the radius k; the attracted point at the vertex p.

We saw, in *Prop.* 2, *Cor.* 1, that the action of a rhombus, on a point placed perpendicularly over its centre, is $A = 4 \text{ arc } \left(\text{tang.} = \frac{1}{r}\sqrt{1 + \frac{1+r^2}{a^2} b^2}\right) + 4 \text{ arc } \left(\text{tang.} = \frac{1}{r'}\sqrt{1 + \frac{1+r'^2}{a^2} b^2}\right) - 2\pi$; in which we must put x^2 for a^2, $2kx - x^2$ for b^2, and we get, for the attraction of the solid,

$$A = 4\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{1}{rx}\sqrt{2k(1+r^2)x - r^2x^2}\right) + 4\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{1}{r'x}\sqrt{2k(1+r'^2)x - r'^2x^2}\right) - 2\pi x.$$

These fluents being exactly similar to the one in *Prop.* 24, if we put $A = \text{arc}\ (\text{tang.} = \frac{1}{rx}\sqrt{2k(1+r^2)x - r^2x^2})$

$$A' = \text{arc}\ (\text{tang.} = \frac{1}{r'x}\sqrt{2k(1+r'^2)x - r'^2x^2})$$

it is easy to see that

$$A = \left\{\frac{2k}{r^2} + 2(k-x)\right\}(\pi - 2A) - \frac{2}{r}\sqrt{2k(1+r^2)x - r^2x^2}$$
$$+\ 2\pi x$$
$$+ \left\{\frac{2k}{r'^2} + 2(k-x)\right\}(\pi - 2A') - \frac{2}{r'}\sqrt{2k(1+r'^2)x - r'^2x^2}$$
$$+\ \text{corr.}$$

If the fluent is to begin when $x = o$, no correction is necessary; for at that term $A = A' = \frac{\pi}{2}$.

When $x = 2k$, $A = \text{arc}\ (\text{tang.} = \frac{1}{r})$, $A' = \text{arc}\ (\text{tang.} = \frac{1}{r'})$ and

$$A = \left\{\frac{2k(1-r^2)}{r^2}\right\}\left\{\pi - 2\ \text{arc}\ (\text{tang.} = \frac{1}{r})\right\} - \frac{4k}{r} + 4k\pi$$
$$+ \left\{\frac{2k(1-r'^2)}{r'^2}\right\}\left\{\pi - 2\ \text{arc}\ (\text{tang.} = \frac{1}{r'})\right\} - \frac{4k}{r'}.$$

If we thought proper, this might still be put under a different form; for $r' = \frac{1}{r}$, and the arcs the complements of each other; and $\pi - 2\ \text{arc}\ (\text{tang.} = \frac{1}{r}) = 2\ \text{arc}\ (\text{tang.} = r)$; also $\pi - 2\ \text{arc}\ (\text{tang.} = \frac{1}{r'}) = 2\ \text{arc}\ (\text{tang.} = r')$.

Cor. 1. When the rhombus is a square, $r = r' = 1$; and the action becomes $A = 4k\pi - 8k$, as we found in *Prop.* 24.

Cor. 2. Let r' be infinite, then $r = o$, and the solid becomes an infinitely long circular cylinder; and it is easy to see that the value of A is reduced to $2k\pi$, as we found before in a different manner.

The foregoing problems, which I have chosen from a great variety that occurred to me, are sufficient to shew the use that may be made of the expressions given in the first section.

The attractions of certain infinitely long cylinders, which were derived, as corollaries, from some of the preceding propositions, present us with several curious relations; with these I shall terminate the present division of my subject.

Let pout, fig. 18, represent the base, or section, of a circular cylinder, infinitely extended both above and below the plane of the figure. Let p be an attracted point in the circumference of the section. Draw the diameter pu, and, at right angles to it, the diameter ot.

By *Cor.* 1, *Prop.* 27, the action of the whole cylinder on the point p is $2k\pi$ (k being the radius of the circular section); the action of that half of the cylinder, whose base is the semicircle opto, is $2k(\pi - 1)$; the action of the other half of the cylinder, which is furthest from p, is $2k$: therefore,

1. The attraction of a sphere is to that of an infinite circular cylinder of the same diameter (on a point at the surface of each) as $\frac{2}{3}$ to 1, which is the ratio of the solidity of a sphere to that of its circumscribing cylinder.

2. The attraction of the whole infinite cylinder, on p, is to the attraction of that half which is furthest from that point, as the circumference of a circle is to its diameter.

3. Consequently, the attraction of the nearest half, is to that of the furthest half, as the difference between the circumference and diameter of a circle is to the diameter; or *nearly* as 2 to 1.

4. In the circle optu, fig. 18, inscribe the parabola owpvt, whose equation is $kx = y^2$, so that its vertex may be at p, and its axis coincide with pu: this parabola will plainly cut the

circle at the quadrantal points o and t; and, I say, that the action, on the point p, of the infinitely long cylinder, whose base is the parabolic area owpvtco, is to the attraction of the furthest half of the infinite circular cylinder, *exactly* as 2 to 1. For the latter action has been shewn to be $2k$; and if in the expression, obtained in *Cor.* 2, *Prop.* 27, we make α and x both $= k$, it is reduced to $A = 4k$.

5. In fig. 18, draw fug perpendicular to pu at u; and from p, through o and t, the lines pof, ptg. The attraction, on the point p, of the infinitely long prism whose base is the triangle pfg, is equal to the attraction of the infinitely long circular cylinder. For the action of the prism is $4 \times \text{pu} \times$ arc. fpu (by *Prop.* 20) $= 8k \times \frac{\pi}{4} = 2k\pi$; and this has been already shewn to be the attraction of the circular solid.

§. V.

Of Solids of greatest Attraction.

The subject of this section has occupied the attention of Mr. PLAYFAIR,* in the same paper I have before noticed; it had previously been treated of by SILVABELLE. FRISI also, in the third volume of his works, gives a solution of the same problem as that which is first considered by Mr. PLAYFAIR, but his result is an erroneous one. None of these writers have pursued the matter any further than what relates to the figure of a homogeneous solid of revolution. My manner of treating the subject connects it intimately with the preceding parts of this paper; otherwise I should not have given the following problems a place here.

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* The problems which I investigate are similar to *the first* of Mr. PLAYFAIR's, where the *equation of a curve* is sought; nor do I at all meddle with that other class of problems which he treats of in the subsequent part of the paper.

Prop. 31.

Suppose that a given quantity of matter is to be formed into a right cylinder of the length $2d$; what must be the figure of its base, so that it shall attract, with the greatest force possible, a point in its surface, and in the middle with respect to its two ends?

Let fig. 19 represent a section of the cylinder, at the attracted point p, parallel to its base. It is plain enough, that, whatever is the nature of the curve pab, we may draw a line pb from p, which shall divide the area into two equal and similar portions pabp, pcbp.

Put the absciss pd $= x$, the ordinate ad $= y$: the mass of the cylinder is $4d\int y\dot{x}$; and, by *Prop.* 4, its attraction on p is

$$4\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{dy}{x\sqrt{d^2+x^2+y^2}}\right).$$

Let C be a constant quantity, and we have only to make the fluxion of the following expression, with respect to y, equal to nothing,* viz.

$$\text{Arc } \left(\text{tang.} = \frac{dy}{x\sqrt{d^2+x^2+y^2}}\right) + Cdy:$$

this gives

$$\frac{x}{(x^2+y^2)\sqrt{d^2+x^2+y^2}} + C = o, \text{or } x^2 - C^2)x^2+y^2)^2(d^2+x^2+y^2) = o,$$

for the equation of the curve pab. Make $y = o$, and let a be the corresponding value of x; the equation becomes

$1 - C^2a^2(d^2+a^2) = o$, whence $C^2 = \frac{1}{a(d^2+a^2)}$; by substituting which value in the equation of the curve, it is ultimately

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* EULERI "Methodus, &c." p. 42 and 185-6-7.

$$a^2(d^2+a^2)x^2-(x^2+y^2)^2(d^2+x^2+y^2)=o.$$

It is proper to remark, that though, in the enunciation, I spoke of the point as being in the surface of the cylinder, yet there is nothing in the above method of investigation that supposes it to be in contact with the solid: if there is to be a given distance between them, the nature of the curve will be the same.

Cor. 1. If we make d infinitely small, there results $a^4x^2-(x^2+y^2)^3=o$ for the equation of the curve bounding *the plane of greatest attraction;* and it is evident, that, by the revolution of this curve about its axis, will be generated *the solid of greatest attraction*, when it is sought for without any such conditions or restrictions as entered into the preceding problem.

This exactly agrees with the conclusion arrived at by SILVABELLE and Mr. PLAYFAIR.

Cor. 2. If, on the other hand, we make d infinitely great, the equation is reduced to $y^2=ax-x^2$, which is that of a circle whose diameter is a, the attracted point being in the circumference. Therefore,—*of all infinitely long cylinders, having the areas of their bases, or transverse sections, equal, that which has a circle for the circumference of the said base, shall exert the greatest action on a point at its surface.*

Prop. 32.

Let a given quantity of matter be fashioned into such a solid as was treated of at the beginning of the last section (in Propositions 24, 25, 26), viz. having its section perpendicular to the axis a regular polygon. The polygon being given in species, it is required to determine the nature of the curve pr, fig. 15, so that the solid may have the greatest possible action on a point at its vertex p.

If we put x for the distance of the generating polygon from the vertex, and y for the perpendicular let fall from the centre of the polygon on one of its sides, the action of the solid is

$$2n\int \dot{x} \text{ arc } \left(\text{tang.} = \frac{1}{r}\sqrt{1+\frac{1+r^2}{x^2}y^2}\right) - \int (n-2)\,\pi\dot{x}$$

by *Prop.* 5, and the mass of the solid is $nr\int y^2\dot{x}$; so that the quantity whose fluxion, with respect to y, must $= o$, is

$2n$ arc $\left(\text{tang.} = \frac{1}{r}\sqrt{1+\frac{1+r^2}{x^2}y^2}\right) + Cnry^2$, and we get $\frac{x}{(x^2+y^2)\sqrt{x^2+(1+r^2)y^2}} + C = o$, or $x^2 - C^2(x^2+y^2)^2(x^2 + (1+r^2)y^2) = o$.

Let a be the value of x when $y = o$, then $C^2 = \frac{1}{a^4}$, and the equation of the curve becomes

$$a^4x^2 - (x^2+y^2)^2\left\{x^2+(1+r^2)y^2\right\} = o.$$

When the polygon is a circle, $r = o$, and the equation is reduced to $a^4x^2 - (x^2+y^2)^3 = o$, the same as we found in *Cor.* 1, *Prop.* 31.

Lemma 1.

To find the attraction of the right prism whose base is the triangle mrv, fig. 1, and height $\dot{d}$, on the point p, in the direction pm; on the supposition, that the density at the ordinate ks is as any function of the absciss mk, and distance pm.

If we use the same notation as in *Prop.* 1, and put f (a, T) for the density of a particle any where at the line ks, we shall find, by proceeding, as we did there,

$$A = \dot{d}\int \frac{arf(a,T)\,T\dot{T}}{(a^2+T^2)\,(a^2+(1+r^2)\,T^2)^{\frac{1}{2}}}.$$

Hence the attraction of a prism, whose height is $\dot{a}$, and base a regular polygon of n sides, composed of triangles having such a law of density as was supposed above, will be, on a point placed perpendicularly over its centre,

$$A = 2n\dot{a}\int\frac{arf\,(a,T)\,T\dot{T}}{(a^2+T^2)\,(a^2+(1+r^2)T^2)^{\frac{1}{2}}}.$$

This expression would be easily integrable on various suppositions. Thus we might conceive the density at the ordinate ks to vary as the line ps, drawn from the attracted point to its extremity s; this would be to make $f\,(a,T) = \sqrt{a^2+(1+r^2)T^2}$; whence $A = 2n\dot{a}\int\frac{arT\dot{T}}{a^2+T^2} = rna\dot{a}\left\{L.\,(a^2+T^2) - L.\,a^2\right\}$.

Again, we might suppose $f\,(a,T) = a^2 + T^2 = (\text{pk})^2$; this would give $A = 2n\dot{a}\int\frac{arT\dot{T}}{\sqrt{a^2+(1+r^2)\,T^2}} = \frac{2nra\dot{a}}{1+r^2}\left\{\sqrt{a^2+(1+r^2)T^2} - a\right\}$.

But the kind of problems we are engaged about does not require us to know the value of A itself; its fluxional coefficient with respect to T being alone wanted, and this is *always*

$$\frac{2narf\,(a,T)\,T}{(a^2+T^2)\,\sqrt{a^2+(1+r^2)\,T^2}} \times \dot{a} \text{ ....} \ (1).$$

For suppose we had actually found the fluent; when we make use of it in such a problem as the last, we must change T into y, and take the fluxion with respect to y, and the result must necessarily be $\frac{2narf\,(a,y)\,y}{(a^2+y^2)\,\sqrt{a^2+(1+r^2)\,y^2}} \times \dot{a}$; which we might have arrived at simply by changing T into y in the expression marked (1).

Lemma 2.

To find the quantity of matter in a right prism, whose base is the triangle rmv, fig. 1, and height $\dot{a}$; supposing the density at any ordinate ks to be $f\,(a,T)$.

The *magnitude* of the element of the prism is $\dot{a}Tr\dot{T}$, and the *mass* of this element is $r\dot{a} \times \mathrm{f}\,(a,T)\ T\dot{T}$; whence the mass of the whole prism is $r\dot{a}\int \mathrm{f}\,(a,T)\ T\dot{T}$.

The mass of a prism whose height is $\dot{a}$ and base a regular polygon of n sides, formed of triangles having this law of density, is $2nr\dot{a}\int \mathrm{f}\,(a,T)\ T\dot{T}$ and its fluxional coefficient, with respect to T, is $2nr\,\mathrm{f}\,(a,T)\ T \times \dot{a}$.

Prop. 33.

Let the last proposition be again proposed, but with this difference, that the solid, instead of being homogeneous, is to be formed of polygonal prismatic elements, having such a law of density as in the preceding lemmas.

By proceeding as before, we shall have for the equation of the curve pr, fig. 15,

$$\frac{2nxr\,\mathrm{f}\,(x,y)y}{(x^2+y^2)\sqrt{x^2+(1+r^2)y^2}} + \mathrm{C}2nr\,\mathrm{f}\,(x,y)\,y = o,$$

or $\frac{x}{(x^2+y^2)\sqrt{x^2+(1+r^2)y^2}} + \mathrm{C} = o$; *which is exactly the same equation as when the solid was supposed to be homogeneous.*

When $r = o$, we have, as before, $a^4x^2 = (x^2+y^2)^3$; which shews that *the result of Mr.* PLAYFAIR *extends to an infinity of cases besides that of homogeneity.*

When, as in our last supposition, $r = o$, and the mass is a solid of revolution, the function $\mathrm{f}\,(a,T)$ expressing the density, is a function of the perpendicular let fall from any particle on the axis of the solid, and of the distance between the foot of that perpendicular and the attracted point.

Scholium.

If the preceding lemmas had been treated on the supposition that the density was variable *along* the line ks, fig. 1, (which is the same as making the density f (t), or more generally f (a, T, t) a function of a, T, and t) their application to the problem we have been considering, would give an indefinite number of different equations, for the curve pr, fig. 15, according to the nature of the assumed function f (a, T, t): every one of which equations will, however, have this peculiarity, that if we make $r = o$, it will become $a^4x^2 = (x^2 + y^2)^3$. For when $r = o$, $t = o$, and f (a, T, t) becomes a function of a and T only, and the case enters into *Prop.* 33 just now considered.

It may be worth while to see an example of this; we should have had, in general, for the action of the polygonal prismatic element of the solid, by *Prop.* 1,

$$A = 2n\dot{a}\iint \frac{a\,\mathrm{f}(a, T, t)\,\dot{T}\dot{t}}{(a^2 + T^2 + t^2)^{\frac{3}{2}}};$$

and the mass of the same element would have been

$$2n\dot{a}\iint \mathrm{f}\,(a, T, t)\,\dot{T}\dot{t}.$$

These must be integrated, with respect to t, from $t = o$ to $t = rT$: which cannot be done till we assign a form for the function f (a, T, t). Let this be $a^2 + T^2 + t^2$, that is to say, let the density at any point q, in the triangle vrm, be as the square of its distance pq from the attracted point p. This will give

$$A = 2n\dot{a}\iint \frac{a\dot{T}\dot{t}}{(a^2 + T^2 + t^2)^{\frac{1}{2}}} = 2na\dot{a}\int \left\{ \mathrm{L}\left(rT + \sqrt{a^2 + (1 + r^2)\,T^2}\right) - \mathrm{L}\sqrt{a^2 + T^2} \right\} \dot{T};$$ and for the mass

$$2n\dot{a}\iint (a^2 + T^2 + t^2)\,\dot{T}\dot{t} = 2n\dot{a}\int \left(a^2 rT + rT^3 + \frac{r^3T^3}{3}\right)\dot{T}.$$

If therefore we solve *Prop.* 32, on this supposition of density, we have for the equation of the curve pr, fig. 15, when the solid has the greatest attraction,

$$x\mathrm{L}\left(ry\sqrt{x^2 + (1 + r^2)y^2}\right) - x\mathrm{L}\sqrt{x^2 + y^2} + \mathrm{C}\left(x^2ry + ry^3 + \frac{r^3y^3}{3}\right) = o.$$

Now, when r is infinitely small, we shall have, by neglecting all the higher powers thereof,

$$\mathrm{L}\left(ry + \sqrt{x^2 + (1 + r^2)y^2}\right) = \mathrm{L}\sqrt{x^2 + y^2} + \frac{ry}{\sqrt{x^2+y^2}};$$

by substituting which our equation becomes

$\frac{x}{\sqrt{x^2+y^2}} + \mathrm{C}\,(x^2 + y^2)$, or $a^4x^2 - (x^2 + y^2)^3 = o$, as we shewed *a priori* must necessarily happen.

I shall just remark here, that, as the results of *Prop.* 32, are not altered by conceiving the density any function of a and T, such is also the case with respect to Problem 31, if T there represent the distance of any particle from a plane passing through the attracted point and the axis of the cylinder. This the reader may easily convince himself of.

The proposition just mentioned (31) is only a particular case of the following very general one.

Prop. 34.

Let uvv′u′, fig. 15, be a rectangle, whose plane is perpendicular to the line pm, and its centre in that line. Let this rectangle move parallel to itself, in the direction pm, and vary in such a manner, that the middle points r and r′ of its sides may continually touch two different curves.

The quantity of matter in the solid so generated being given, and the nature of one of the curves as pr′, to find what must be the other curve pr, so that the action of the solid, on a point at its vertex p, may be the greatest possible.

Put x for the absciss pm, y' and y for the ordinates mr′, mr; then by *Prop.* 4, the action of the solid will be

$4\int \dot{x}$ arc $\left(\text{tang.} = \frac{yy'}{x\sqrt{x^2+y^2+y'^2}}\right)$; and its mass is $4\int yy'\dot{x}$.

But y' is a given function of x, suppose $\text{f}(x)$. The quantity

$$\text{Arc}\left(\text{tang.} = \frac{y\text{f}(x)}{x\sqrt{x^2+y^2+\text{f}(x)^2}}\right) + Cy\text{f}(x)$$

is therefore to have its fluxion, with respect to y, made $= o$: and this gives, for the equation of the curve pr,

$\frac{x}{(x^2+y^2)\sqrt{x^2+y^2+\text{f}(x)^2}} + C = o$, or $x^2 - C^2(x^2+y^2)^2(x^2+y^2+\text{f}(x)^2) = o$.

Ex. 1. Let $\text{f}(x) = ax$, or pr′ be a straight line, the equation of pr must be $x^2 - C^2(x^2+y^2)^2\left\{(1+a^2)x^2+y^2\right\} = o$.

Ex. 2. Let pr′ be a circle, or $\text{f}(x)^2 = 2kx - x^2$, k being the radius, then $x^2 - C^2(x^2+y^2)^2(2kx+y^2) = o$, is the equation of the other curve.

Ex. 3. If pr′ is a parabola, or $\text{f}(x)^2 = ax$, the equation of pr is $x^2 - C^2(x^2+y^2)^2(ax+x^2+y^2) = o$.

Scholium.

In *Prop.* 27, after having found the action of the solid there treated of, we derived, as corollaries, the action of parabolic and circular cylinders of infinite length, by separately making infinite the diameter of the circle and the parameter of the parabola. Perhaps it might therefore be supposed, that if we made k infinite in the second of the preceding examples, or a

infinite in the third, the result would be the equation of the base of the infinitely long *cylinder* of greatest attraction; which however is by no means the case; for that was found to be a circle, whereas the equation we get here is

$$x - C'(x^2 + y^2) = o,$$

and if we make a infinitely great in the first example, the equation becomes $C' = x^2 + y^2$, or the line pr is a circle with its centre at the attracted point.

We might resolve this problem, on a variety of hypotheses respecting the density; or we might add other conditions of a different kind: for instance, not only the mass of the solid, but the area of the section, passing through the required curve pr and axis pm, might be supposed constant. But I pass on to other suppositions respecting the force of attraction; which will be treated with as much brevity as possible.

Lemma 3.

To find the attraction of the triangle vrm, fig. 1, on the point p, in the direction pm, supposing the force to be inversely as the mth power of the distance.

Keeping the same notation as in *Prop.* 1, we have, for the attraction of an element at q, $\frac{\dot{T}\dot{t}}{(a^2+T^2+t^2)^{\frac{m}{2}}}$; which being resolved, gives, for the force of the whole triangle, in the direction pm, $A = \int\int \frac{a\dot{T}\dot{t}}{(a^2+T^2+t^2)^{\frac{m+1}{2}}}$; the fluent is to be taken from $t = o$ to $t = rT$, and we have

$$A = \int \frac{arT\dot{T}}{(a^2+T^2)(a^2+(1+r^2)T^2)^{\frac{m-1}{2}}} \Big\{ 1 - (2-m)\frac{r^2T^2}{3(a^2+T^2)} + (2-m)(4-m)\frac{r^4T^4}{3.5(a^2+T^2)^2} - \&c. \Big\}.$$

The further integration, with respect to T, is not necessary for our purpose.

Cor. 1. If we multiply this by $2n$, it will be the attraction of a regular polygon of n sides; and making n infinitely great and r infinitely small, the attraction of a circle to the radius T is found to be

$$\mathrm{A} = \int \frac{2nraT\dot{T}}{(a^2+T^2)^{\frac{m+1}{2}}} = \frac{-2nra}{(m-1)(a^2+T^2)^{\frac{m-1}{2}}} + \frac{2nra}{(m-1)a^{m-1}}$$

which, by putting π for nr, is

$$\mathrm{A} = 2\pi \left\{ \frac{1}{(m-1)a^{m-2}} - \frac{a}{(m-1)(a^2+T^2)^{\frac{m-1}{2}}} \right\}$$

the same as is found differently by other writers.

Cor. 2. When r becomes infinite, the triangle vrm is changed into a parallelogram, infinitely extended in the direction rv; and we have

$$\mathrm{A} = \int \frac{a\dot{T}}{(a^2+T^2)(rT)^{m-2}} \left\{ 1 - (2-m)\frac{r^2T^2}{3(a^2+T^2)} + (2-m)(4-m)\frac{r^4T^4}{3 \cdot 5\,(a^2+T^2)^2} - \&c. \right\}$$

which, when m is an even positive whole number greater than 2, is reduced to $\mathrm{A} = \frac{2.4.6 \ldots\ldots (m-2)}{3 \cdot 5 \cdot 7 \ldots\ldots (m-1)} \int \frac{a\dot{T}}{(a^2+T^2)^{\frac{m}{2}}}$.

Cor. 3. If instead of the action of the triangle vrm, that of a rectangle, whose sides are rm (y) and rv (y'), had been required, we must have proceeded exactly in the same manner, but the fluent, with respect to t, must have been taken from $t = o$, to $t = y'$; so that we have only to substitute y' for rT, in the value found by the lemma, and there results

$$\mathrm{A} = \int \frac{ay\dot{T}}{(a^2+T^2)(a^2+T^2+y'^2)^{\frac{m-1}{2}}} \left\{ 1 - (2-m)\frac{y'^2}{3(a^2+T^2)} + (2-m)(4-m)\frac{y'^4}{3 \cdot 5\,(a^2+T^2)^2} - \&c. \right\}$$

Another Method of finding the Action of the Triangle vrm.

The expressions we have found will terminate only when m is one of the series of numbers 2, 4, 6, &c. If m is among the odd numbers 1, 3, 5, &c. $\frac{m+1}{2}$ will be a whole positive number; and we have for the action of the triangle, or rectangle (accordingly as the fluent, with respect to t, is taken from $t=o$, to $t=rT$, or from $t=o$, to $t=y'$) provided m is greater than 3,

$$A = \int a\dot{T}\left\{\frac{1}{(m-1)(a^2+T^2)} \times \frac{t}{(a^2+T^2+t^2)^{\frac{m-1}{2}}} + \frac{m-2}{(m-1)(m-3)(a^2+T^2)^2} \times \frac{t}{(a^2+T^2+t^2)^{\frac{m-3}{2}}} + \frac{(m-2)(m-4)}{(m-1)(m-3)(m-5)(a^2+T^2)^3} \times \frac{t}{(a^2+T^2+t^2)^{\frac{m-5}{2}}} + \ldots\ldots + \frac{(m-2)(m-4)\ldots\ldots 3}{(m-1)(m-3)(m-5)\ldots\ldots 2(a^2+T^2)^{\frac{m-1}{2}}} \times \frac{t}{a^2+T^2+t^2} + \frac{(m-2)(m-4)\ldots\ldots 3}{(m-1)(m-3)(m-5)\ldots\ldots 2(a^2+T^2)^{\frac{m-1}{2}}} \times \frac{1}{\sqrt{a^2+T^2}} \times \text{arc}\left(\text{tang.} = \frac{t}{\sqrt{a^2+T^2}}\right)\right\}.$$

Let us, for brevity, denote this quantity by $\int a\dot{T}\varphi\,(a, m, T, t)$; then for the action of the triangle vrm we have $A = \int a\dot{T}\varphi\,(a, m, T, rT)$; and for the rectangle, whose sides are rm (y) and rv (y') $A = \int a\dot{T}\varphi\,(a, m, T, y')$. When $m = 1$, or 3, the above expression will not give the attraction; but we evidently have, in the case of $m = 1$,

$$A = \int \frac{a\dot{T}}{\sqrt{a^2+T^2}} \times \text{arc}\left(\text{tang.} = \frac{t}{\sqrt{a^2+T^2}}\right);$$ and when $m = 3$,

$$A = \int a\dot{T}\left\{\frac{1}{2(a^2+T^2)} \times \frac{t}{a^2+T^2+t^2)} + \frac{1}{2(a^2+T^2)^{\frac{3}{2}}} \times \text{arc}\left(\text{tang.} = \frac{}{\sqrt{a^2+T^2}}\right)\right\}.$$

Cor. 1. For a polygon of n sides, these expressions must be multiplied by $2n$ as usual; and when m is greater than 3, $A = 2n\int a\dot{T}\phi\,(a, m, T, rT)$, if in this we make n infinitely great and r infinitely small, it ought to enter into the general case of the attraction of a circle given in *Cor.* 1, to the first part of the lemma: and in fact we get

$$A = \left\{\frac{1}{m-1} + \frac{m-2}{(m-1((m-3)} + \frac{(m-2)(m-4)}{(m-1)(m-3)\,m-5)} + \ldots\ldots\ldots\ldots + \frac{(m-2)(m-4)\ldots\ldots 3}{(m-1)(m-3)\ldots\ldots 2} + \frac{(m-2)(m-4)\ldots\ldots\ldots 3}{(m-1)(m-3)\ldots\ldots\ldots 2}\right\} \times \int \frac{2narT\dot{T}}{(a^2+T^2)^{\frac{m+1}{2}}},$$

or, because the quantity between the brackets is plainly equal to unity, becomes $A = \int \frac{2nraT\dot{T}}{(a^2+T^2)^{\frac{m+1}{2}}}$ which is the same form as was found before for the general case.

Cor. 2. When r becomes infinite, and the triangle rmv is changed into an infinitely extended rectangle, we have for its attraction

$$A = \frac{3.5.7\ldots\ldots\ldots(m-2)}{2.4.6\ldots\ldots\ldots(m-1)} \int \frac{a\pi\dot{T}}{2\,(a^2+T^2)^{\frac{m}{2}}},$$

except when $m = 1$, in which case, $A = \int \frac{a\pi\dot{T}}{2\sqrt{a^2+T^2}}$.

Scholium.

This lemma has been treated on the supposition that the density is the same at every part of the triangle rmv, fig. 1; but there are other hypotheses which render the solution easier: for instance, we may conceive the density of a particle at q to be as its distance (t) from the line rm, in which case

$$A = \iint \frac{a\dot{T}t\dot{t}}{(a^2+T^2+t^2)^{\frac{m+1}{2}}} = \int \left\{\frac{-a\dot{T}}{(m-1)(a^2+T^2+t^2)^{\frac{m-1}{2}}} + \frac{a\dot{T}}{(m-1)(a^2+T^2)^{\frac{m-1}{2}}}\right\}$$

where t must be made rT or y' accordingly as the action of a triangle or rectangle is required.

From this simple case, we may not only arrive at some curious results, connected with the particular hypothesis of density, but may find with equal ease the figure of a *homogeneous* solid of revolution of greatest attraction, as will just now be seen.

If the density was to be a function of a and T only, it would be sufficient to multiply the values found in the lemma, by that function, see lemma 1.

Prop. 35.

Let *Prop.* 31 be again proposed, but with this difference, that the force is inversely as the mth power of the distance, and that the density of any particle of the cylinder is as its distance (t) from that middle section (parallel to the ends of the cylinder) which passes through the attracted point.

In the expression we just now found, in the preceding scholium, put x for a, and d (half the length of the cylinder) for t. The action of the cylinder is

$$\mathrm{A} = 4\iint \left\{ \frac{-x\dot{x}\dot{T}}{(m-1)(x^2+T^2+d^2)^{\frac{m-1}{2}}} + \frac{x\dot{x}\dot{T}}{(m-1)(x^2+T^2)^{\frac{m-1}{2}}} \right\};$$

its quantity of matter is $4\iiint \dot{x}\dot{T}\dot{t}t = 2\iint \dot{x}\dot{T}t^2$; so that we have, for the equation of the curve bounding the base,

$$\frac{x}{(x^2+y^2)^{\frac{m-1}{2}}} - \frac{x}{(x^2+y^2+d^2)^{\frac{m-1}{2}}} + \mathrm{C}d^2 = o.$$

Cor. 1. When d is infinitely small, this becomes

$$\frac{m-1}{2} \times \frac{xd^2}{(x^2+y^2)^{\frac{m+1}{2}}} + \mathrm{C}d^2, \text{ or } \frac{x}{(x^2+y^2)^{\frac{m+1}{2}}} + \mathrm{C}' = o.$$

Let a be the value of x when $y = o$, then $\mathrm{C}' = -\frac{1}{a^m}$; and the equation of the curve, bounding *the plane of greatest attraction*, is

$$a^m x = (x^2 + y^2)^{\frac{m+1}{2}},$$

which is exactly the same result as that obtained by Mr. PLAYFAIR, p. 203, on the supposition of *homogeneity*; and this was to be expected; for, though a certain condition of the density of the cylinder entered into the foregoing problem, yet when *d* vanishes, and the solid becomes a plane, we must evidently obtain the same result as if it had been arrived at by supposing the cylinder homogeneous; which in fact it will be when the length is evanescent.

Nor is this observation to be confined to that particular case when the density is as t: if we had solved the problem on the supposition of any function of x, T, and t, for the density, it is easy to see that though different functions will give different results when d is finite, yet when the solid becomes *a plane*, and $d = o$, the equation will always be reduced to

$$a^m x = (x^2 + y^2)^{\frac{m+1}{2}}.$$

Hence we may conclude, that, *the solid of revolution which shall exercise the greatest attraction on a point in its axis, when the force is inversely as the mth power of the distance, and the density either uniform, or any function whatever of x and T (T being the perpendicular let fall from any particle to the axis of the solid, and x the distance between the foot of that perpendicular and the attracted point) will have, for the equation of its generating curve,*

$$a^m x = (x^2 + y^2)^{\frac{m+1}{2}}.$$

Cor. 2. Nothing can be learned from the equation

$$\frac{x}{(x^2+y^2)^{\frac{m-1}{2}}} - \frac{x}{(x^2+y^2+d^2)^{\frac{m-1}{2}}} + Cd^2 = o,$$

when $m = 1$. The curve is then transcendent, and has for its equation $x\mathrm{L}.(x^2 + y^2 + d^2) - x\mathrm{L}.(x^2 + y^2) + \mathrm{C}d^2 = 0$.

Cor. 3. If the cylinder becomes infinitely long, (m being positive and greater than unity) the equation of its base is

$$\frac{x}{(x^2+y^2)^{\frac{m-1}{2}}} + \mathrm{C}' = 0;$$

let a be the value of x when $y = 0$; then $\mathrm{C}' = \frac{-1}{a^{m-2}}$, and the equation becomes

$$\frac{x}{(x^2+y^2)^{\frac{m-1}{2}}} - \frac{1}{a^{m-2}} = 0.$$

If $m = 2$, as in the case of nature, this becomes $\frac{x^2}{x^2+y^2} = 1$, *so that the infinitely long cylinder of greatest attraction will be an infinitely long rectangle, with its edge turned to the attracted point.*

If $m = 3$, we have $ax = x^2 + y^2$, the equation of a circle with the attracted point in its circumference.

If $m = 4$, the equation is $a^2x = (x^2 + y^2)^{\frac{3}{2}}$, which is Mr. Playfair's *curve of equal attraction.*

If we want the figure of the infinite cylinder of greatest attraction, when $m = 1$, we must have recourse to the last corollary; where we found

$$x\mathrm{L}\,(x^2 + y^2 + d^2) - x\mathrm{L}\,(x^2 + y^2) = \mathrm{C}'.$$

This, when d is infinite gives $x\mathrm{L}.d^2 = \mathrm{C}'$, or, $x =$ const., *the equation of a plane perpendicular to the axis of x.*

Cor. 4. If we would solve *Proposition* 34, but with this difference, that the force is now inversely as the mth power of the distance, and the density, in the generating rectangle uvv′u′, fig. 15, is, at any point, as its distance from rm or y; we need only put f(x) (given by the nature of the curve pr) for d, in the equation here found, and we get that of pr, in the case of greatest attraction: viz.

$$\frac{x}{(x^2+y^2)^{\frac{m-1}{2}}} - \frac{x}{(x^2+y^2+f(x)^2)^{\frac{m-1}{2}}} + Cf(x)^n = o.$$

Prop. 36.

To solve *Prop.* 32, the force being supposed inversely as the *m*th power of the distance, and the generating polygon being composed of triangles having such a law of density as that in the scholium to lemma 3.

By using the value found in that scholium, and proceeding, in other respects, as in the similar propositions already given, we find, for the equation of the curve touching the sides of the polygon,

$$\frac{x}{(x^2+y^2)^{\frac{m-1}{2}}} - \frac{x}{(x^2+(1+r^2)y^2)^{\frac{m-1}{2}}} + Cr^n y^n = o.$$

Prop. 37.

Let *Prop.* 32 be yet once more resolved, on the supposition that the force is inversely as the *m*th power of the distance; and the density, in the triangles forming the generating polygon, either uniform, or as any function of x and T.

If we make use of the first value of A in lemma 3, we get, for the equation of the curve touching the sides of the polygon,

$$\frac{x}{(x^2+y^2)(x^2+(1+r^2)y^2)^{\frac{m-1}{2}}}\left\{1-(2-m)\frac{r^2y^2}{3(x^2+y^2)}+(2-m)(4-m)\frac{r^4y^4}{3\cdot 5(x^2+y^2)^2}-\&c.\right\}+C=o.$$

When $r = o$, or the polygon becomes a circle, this equation is reduced to $\frac{x}{(x^2+y^2)^{\frac{m+1}{2}}} + C = o$, as was found in another manner, in *Cor.* 1, *Prop.* 35.

If r is finite, the above expression will terminate when m is a whole positive even number; and consequently the guiding curve will then be algebraic. But, if m be amongst the numbers 5, 7, 9, 11, &c., we must use the other expression found in the lemma, and there arises, for the guiding curve, the transcendent equation

$$x\phi\,(x, m, y, ry) + Cry = o.$$

If $m = 1$, the equation is

$$\frac{x}{\sqrt{x^2+y^2}} \times \text{arc}\ \left(\text{tang.} = \frac{ry}{\sqrt{x^2+y^2}}\right) + Cry = o;$$ and, finally, when $m = 3$,

$$\frac{x}{x^2+y^2} \times \frac{ry}{x^2+(1+r^2)y^2} + \frac{x}{(x^2+y^2)^{\frac{3}{2}}}\,\text{arc}\ \left(\text{tang.} = \frac{ry}{\sqrt{x^2+y^2}}\right) + Cry = o.$$

In like manner, might be solved *Prop.* 31 and 34, the force and density being as in Lemma 3, but this I leave to the reader.

Prop. 38.

The force being inversely as the mth power of the distance (where m is any whole positive number), and the density either uniform or any function of x and T,* the base of the infinitely long cylinder of greatest attraction has, for its equation,

$$\frac{x}{(x^2+y^2)^{\frac{m}{2}}} + C = o;$$

for it will appear from lemma 3, and its corollaries, that, whether m be odd or even (that is to say when it is any number in the series 1, 2, 3, 4, 5, &c.), the attraction of an infinite cylinder will be of the form

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* What this means with respect to a cylinder, is shewn at the end of the scholium to *Prop.* 33; and with respect to a solid of revolution in *Prop.* 33.

$A = D\iint \frac{x f(x, T)\, \dot{T} \dot{x}}{(x^2 + T^2)^{\frac{m}{2}}}$, D being a function of *m*; hence the truth of the proposition is manifest. And because the equation of the curve generating the solid of revolution of greatest attraction (on the same hypotheses of force and density) has been shewn to be $\frac{x}{(x^2 + y^2)^{\frac{m+1}{2}}} + C = o$, we have the following remarkable

Theorem.

m being any whole positive number, and the density either uniform or as any function of x and T, the same curve which, by revolving, generates the solid of revolution of greatest attraction, when the force is inversely as the mth power, shall be the base of the infinitely long cylinder of greatest attraction, when the force is inversely as the (m + 1th) power.

Numberless other interesting questions might be proposed, relating to solids of greatest attraction; for instance, we may inquire what must be the curve bounding the base of a cylinder of given mass and length so that it shall exercise the greatest action in a direction *parallel* to its axis.

But as this kind of inquiry proceeds exactly in the same way as the other (only we must use the attraction B, instead of A, in *Prop.* 1), it is unnecessary to lengthen a paper which has already been extended too far.

APPENDIX TO §. III.

Of the Attraction of an infinitely long Prism, whose Base is any right lined Figure whatever.

Prop. A.

Let the rectangle bb′ c′c, fig. 20, be the section or base of a prism, infinitely extended on both sides of it, and let the line psu bisect the opposite sides bb′, cc′ of the rectangle.

It is required to find the attraction of the infinitely long solid, on the point p, in the direction psu.

Let C be the centre of the rectangle, put $k = s\mathrm{C}$, $a = \mathrm{bs}$, $u = \mathrm{pC}$; draw rm perpendicular to $s\mathrm{C}u$, and put $x = \mathrm{Cm}$. Now it appears, from *Cor.* 2, *Prop.* 1 of the paper (putting A for the required attraction) that

$\mathrm{A} = 4\int \dot{\overline{\mathrm{pm}}} \times \text{arc}\left(\text{tang.} = \frac{\mathrm{rm}}{\mathrm{pm}}\right) = 4\int \dot{x}\ \text{arc}\left(\text{tang.} = \frac{a}{u+x}\right)$ $= 4x\ \text{arc}\left(\text{tang.} = \frac{a}{u+x}\right) - 4\int x\ \dot{\overline{\text{arc}}}\left(\text{tang.} = \frac{a}{u+x}\right)$ the last term of which is $4\int \frac{ax\dot{x}}{(u+x)^2 + a^2}$; put $u + x = z$, $\dot{x} = \dot{z}$, $x = z - u$, and it becomes $4\int \frac{(az - au)\dot{z}}{a^2 + z^2}$, which is $4a\mathrm{L}.(a^2 + z^2)^{\frac{1}{2}}$ $- 4u\ \text{arc}\left(\text{tang.} = \frac{z}{a}\right)$ so that

$\mathrm{A} = 4x\ \text{arc}\left(\text{tang.} = \frac{a}{u+x}\right) - 4u\ \text{arc}\left(\text{tang.} = \frac{u+x}{a}\right) + 4a\mathrm{L}.$ $(a^2 + (u + x)^2)^{\frac{1}{2}}$, or $\mathrm{A} = 4(x + u)\ \text{arc}\left(\text{tang.} = \frac{a}{u+x}\right) -$ $2u\pi + 4a\mathrm{L}.(a^2 + (u + x)^2)^{\frac{1}{2}}$,

which fluent being taken from $x = -k$ to $x = k$ gives

$\mathrm{A} = 4(u + k)\ \text{arc}\left(\text{tang.} = \frac{a}{u+k}\right) - 4(u - k)\ \text{arc}\left(\text{tang.} = \frac{a}{u-k}\right) + 4a\mathrm{L}.(a^2 + (u + k^2)^{\frac{1}{2}} - 4a\mathrm{L}.(a^2 + (u - k^2)^{\frac{1}{2}}$.

If we choose to express this by the lines and angles of the figure (20), it is

$$A = 4 \times \text{pu} \times \text{arc}, c\text{pu} - 4 \times \text{ps} \times \text{arc}, b\text{ps} + 4 \times \text{bs} \times L\,\frac{\text{pc}}{\text{pb}}.$$

Prop. B.

Let the section of the prism be an isosceles triangle; the attracted point p being in the line psm (fig. 21), which passes through the vertex s to the middle of the base r′p′.

Draw rm parallel to the base, and put $r =$ tang. rsm; call ps, u; sm, x; then rm $= rx$; and we have for the attraction of the infinite solid

$$A = 4\!\int\! \dot{x}\ \text{arc}\ \left(\text{tang.} = \frac{rx}{u+x}\right) = 4x\ \text{arc}\ \left(\text{tang.} = \frac{rx}{u+x}\right) - 4\int \frac{x\left(\frac{rx}{u+x}\right)}{1+\left(\frac{rx}{u+x}\right)^2};$$ the last term is $$-4\int \frac{rx\dot{x}\,(u+x) - rx^2\dot{x}}{(u+x)^2 + r^2x^2} = -4\int \frac{rux\dot{x}}{(u+x)^2 + r^2x^2} = -4\int \frac{rux\dot{x}}{u^2 + 2ux + (1+r^2)\,x^2} = -\frac{4}{1+r^2}\int \frac{rux\dot{x}}{\frac{u^2}{1+r^2} + \frac{2u}{1+r^2}\,x + x^2} = -\frac{4}{1+r^2}\int \frac{rux\dot{x}}{\left(x+\frac{u}{1+r^2}\right)^2 + \left(\frac{ur}{1+r^2}\right)^2} = -\frac{4}{1+r^2}\int \frac{rux\dot{x}}{(x+\alpha)^2 + r^2\alpha^2},$$ if we put $\alpha = \frac{u}{1+r^2}$. Make, moreover $x + \alpha = z$, $x = z - \alpha$, $\dot{x} = \dot{z}$, and it becomes $-\frac{4}{1+r^2}\int \frac{(ruz - ru\alpha)\,\dot{z}}{z^2 + r^2\alpha^2}$, which fluent is

$$-\frac{4}{1+r^2}\left\{ruL\,.\,\sqrt{z^2 + r^2\alpha^2} - u\ \text{arc}\ \left(\text{tang.} = \frac{z}{r\alpha}\right)\right\},$$

we have then at length

$$A = 4x\ \text{arc}\ \left(\text{tang.} = \frac{rx}{u+x}\right) - \frac{4}{1+r^2}\left\{ruL\,\sqrt{(x+\alpha)^2 + r^2\alpha^2} - u\ \text{arc}\ \left(\text{tang.} = \frac{x+\alpha}{r\alpha}\right)\right\} + \text{cor.}$$

Cor. If the position of the attracting solid be reversed, as in fig. 22, call ps, u, and the attraction will be given by the same formula; only the fluent (if it begin at the point) must now be taken from o to $-x$, instead of from o to x. A being a function of r, x, and u, may be represented by $2\varphi\,(r, x, u)$. To correct the fluent, let sm (fig. 23) $=$ X, sm$'=x$, then, the attraction of the solid, whose base is the quadrilateral figure ρrr$'\rho'$, will be $2\varphi\,(r, x, u) - 2\varphi\,(r, \mathrm{X}, u)$.

In figure 24, call ps, u; sm, X; sm$'$, x. The action of the solid, whose base is ρrr$'\rho'$, is expressed by $2\varphi\,(r, -x, u) - 2\varphi\,(r, -\mathrm{X}, u)$.

In fig. 25, put ps $= u$, ps$' = u'$, sm $=$ s$'$m $= x$, tang. of rsm $= r$: the attraction of the solid, whose base is the rhombus srs$'\rho$, on a point p in the produced diameter of the section, is $2\varphi\,(r, x, u) - 2\varphi\,(r, o, u) + 2\varphi\,(r, -x, u') - 2\varphi\,(r, -o, u')$.

Prop. C.

Let fig. 26 represent the base or section of an infinitely long prism, and let this base be any right lined figure whatever, regular or irregular: from p, a point in the same plane, draw *any line* pq, cutting the base at s and m$'''$. It is required to find the action of the solid on the point p, in the direction pq.

From the angles r, r$'$, r$''$, r$'''$, &c. of the base, let fall the perpendiculars rm, r$'$m$'$, r$''$m$''$, r$'''$m$'''$, &c. on the line pq. Prolong the sides of the polygon till they meet pq at the points s, s$'$, s$''$, s$'''$, &c.

Put $u =$ ps, $u' =$ ps$'$, $u'' =$ ps$''$, $u''' =$ ps$'''$, &c.; and

$$x = \mathrm{sm}, \left\{\begin{matrix} x' = \mathrm{s'm'} \\ \mathrm{X'} = \mathrm{s'm} \end{matrix}\right], \left\{\begin{matrix} x'' = \mathrm{s''m''} \\ \mathrm{X''} = \mathrm{s''m'} \end{matrix}\right], \left\{\begin{matrix} x''' = \mathrm{s'''m''} \\ \mathrm{X'''} = \mathrm{s'''m'''} \end{matrix}\right], \text{\&c.}$$

Also, let $r =$ tang. rsm, r$' =$ tang. r$'$s$'$m$'$, r$'' =$ tang. r$''$s$''$m$''$, $r''' =$ tang. r$'''$s$'''$m$'''$, &c. then it appears from the last proposition, that the attraction, of the upper half of the solid, is expressed by

$$\begin{aligned} &\phi(r,\ x,u) - \phi(r,\ o,u) \\ +&\phi(r',\ x',u') - \phi(r',\ X',u') \\ +&\phi(r'',\ x'',u'') - \phi(r'',\ X'',u'') \\ +&\phi(r''',-x''',u''') - \phi(r''',-X''',u''') \\ +&\quad \&c. \quad - \quad \&c. \end{aligned}$$

And in the same manner is found the attraction of the lower portion. If any part of the polygon, as pp', is parallel to pq, the attraction of that portion of the solid may be found by *Prop.* A.

Scholium to Prop. 25, page 273.

The following expression includes the attraction (on a point at the pole or vertex) of all this class of solids, where the generating plane is a regular polygon, and guiding curve a conic section: or where $y^2 = \alpha^2(\beta x + \gamma x^2)$.

$$A = 2n\left(x + \frac{\beta\alpha^2}{1+\gamma\alpha^2}\right) \text{arc}\left(\text{tang.} = \frac{1}{rx}\sqrt{\mu x + \nu x^2}\right) - \left\{(n-2)\right.$$

$$\left. x + \frac{n\beta\alpha^2}{1+\gamma\alpha^2}\right\}\pi + \phi$$

in which $\mu = \beta\alpha^2(1+r^2)$, $\nu = 1 + \gamma\alpha^2(1+r^2)$, and

$$\phi = \frac{2nr\beta\alpha^2}{\sqrt{\nu}\,(1+\gamma\alpha^2)}\,\text{L}\left(\frac{\sqrt{\nu x}}{\sqrt{\mu}} + \sqrt{\frac{\nu x}{\mu} + 1}\right), \text{ or } = \frac{2nr\beta\alpha^2}{\sqrt{-\nu}\,(1+\gamma\alpha^2)}\ \text{arc}$$

$\left(\text{sine} = \frac{\sqrt{-\nu x}}{\sqrt{\mu}}\right)$, accordingly as ν is positive or negative.

Ex. 1. Let $\gamma = o$, $\alpha = 1$, $y = \beta x$; in which case the solid is the polygonal parabolic conoid treated of in the proposition; and we have $\mu = \beta(1+r^2)$, $\nu = 1$, whence

$$A = 2n(x+\beta)\,\text{arc}\left(\text{tang.} = \frac{1}{rx}\sqrt{\beta(1+r^2)x+x^2}\right) - \{(n-2)$$

$$x + n\beta\}\pi + 2nr\beta L\left(\frac{\sqrt{x}}{\sqrt{\beta(1+r^2)}} + \sqrt{\frac{x}{\beta(1+r^2)}+1}\right),$$

the same as was found before.

Ex. 2. Let $\alpha^2 = \frac{b^2}{a^2}$, $\beta = a$, $\gamma = -1$, $y^2 = \frac{b^2}{a^2}(ax - x^2)$: here the curve pr, fig. 15, is an ellipsis whose diameters are a and b, a being that which coincides with the axis pm. We have, in this case, $\mu = \frac{b^2}{a}(1+r^2)$, $\nu = 1 - \frac{b^2}{a^2}(1+r^2)$, and the attraction of a *polygonal spheroid*, on a point at its pole, is

$$A = 2n\left(x + \frac{b^2a}{a^2-b^2}\right)\text{arc}\left(\text{tang.} = \frac{1}{rx}\sqrt{\frac{b^2}{a}(1+r^2)x + \left(1-\frac{b^2}{a^2}(1+r^2)\right)x^2}\right)$$

$$-\left\{(n-2)x + \frac{nb^2a}{a^2-b^2}\right\}\pi + \phi,$$

where $\phi = \frac{2nra^2b^2}{(a^2-b^2)\sqrt{a^2-(1+r^2)b^2}} L\left\{\frac{\sqrt{\{a^2-(1+r^2)b^2\}x}}{\sqrt{b^2a(1+r^2)}} + \sqrt{\frac{\{a^2-(1+r^2)b^2\}x}{b^2a(1+r^2)}+1}\right\}$, or

$$= \frac{2nra^2b^2}{(a^2-b^2)\sqrt{(1+r^2)b^2-a^2}}\,\text{arc}\left(\text{sine} = \sqrt{\frac{\{(1+r^2)b^2-a^2\}x}{b^2a(1+r^2)}}\right),$$

accordingly as $\frac{a^2}{b^2}$ is greater or less than $1+r^2$, or as $\frac{a}{b}$ is greater or less than the secant of half the angle formed at the centre of the generating polygon by one of its sides.

When $x = a$, the first arc in the above expression becomes simply arc $\left(\text{tang.} = \frac{1}{r}\right) = \frac{(n-2)\pi}{2n}$, and we have for the action of the *whole solid*, $A = \psi - \frac{2ab^2}{a^2-b^2}\pi$, ψ representing ϕ after a has been put for x.

In like manner, may the action of the solid be found when the guiding curve is an hyperbola; the only difference between that case, and the one we have just considered, being in the value of γ, which must be taken $+1$ instead of -1.

Scholium to Cor. 3, *Prop.* 27, page 278.

If the variable rectangle is given *in species*, and the touching curves are conic sections; that is, if

$$y^2 = \alpha^2(\beta x + \gamma x^2),\ y'^2 = \alpha'^2(\beta x + \gamma x^2),$$

we shall have, for the action of the generated solid, on a point at its vertex by *Prop.* 4,

$A = 4\int\dot{x}$ arc $\left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1 + r^2)\alpha^2(\beta x + \gamma x^2)}\right) +$
$4\int\dot{x}$ arc $\left(\text{tang.} = \frac{1}{r'x}\sqrt{x^2 + (1 + r'^2)\alpha'^2(\beta x + \gamma x^2)}\right) - 2\pi x,$
where $r = \frac{\alpha'}{\alpha}$, $r' = \frac{\alpha}{\alpha'}$; and by actually taking the fluent,

$A = 4\left(x + \frac{\beta\alpha^2}{1+\gamma\alpha^2}\right)$ arc $\left(\text{tang.} = \frac{1}{rx}\sqrt{\mu x + \nu x^2}\right) - \frac{2\beta\alpha^2}{1+\gamma\alpha^2}\pi + \phi$
$+ 4\left(x + \frac{\beta\alpha'^2}{1+\gamma\alpha'^2}\right)$ arc $\left(\text{tang.} = \frac{1}{r'x}\sqrt{\mu' x + \nu' x^2}\right) - \frac{2\beta\alpha'^2}{1+\gamma\alpha'^2}\pi + \phi'$
$- 2\pi x$, where $\mu = \beta\alpha^2(1+r^2)$, $\nu = 1 + \gamma\alpha^2(1+r^2)$, $\mu' = \beta\alpha'^2$
$(1+r'^2)$, $\nu' = 1 + \gamma\alpha'^2(1+r'^2)$,

$\phi = \frac{4r\beta\alpha^2}{\sqrt{\nu}\,(1+\gamma\alpha^2)}\, \mathrm{L}\left(\frac{\sqrt{\nu x}}{\sqrt{\mu}} + \sqrt{\frac{\nu x}{\mu} + 1}\right)$, or $= \frac{4r\beta\alpha^2}{\sqrt{-\nu}\,(1+\gamma\alpha^2)}$ arc
$\left(\text{sine} = \frac{\sqrt{-\nu x}}{\sqrt{\mu}}\right)$
according as ν is positive or negative,

$\phi' = \frac{4r'\beta\alpha'^2}{\sqrt{\nu'}\,(1+\gamma\alpha'^2)}\, \mathrm{L}\left(\frac{\sqrt{\nu' x}}{\sqrt{\mu'}} + \sqrt{\frac{\nu' x}{\mu'} + 1}\right)$, or $= \frac{4r'\beta\alpha'^2}{\sqrt{-\nu'}\,(1+\gamma\alpha'^2)}$ arc
$\left(\text{sine} = \frac{\sqrt{-\nu' x}}{\sqrt{\mu'}}\right)$ as ν' is positive or negative.

If, in the preceding expression, we make r and α' infinite, and $r' = o$, it is reduced to

$A = 4\left(x + \frac{\beta\alpha^2}{1+\gamma\alpha^2}\right)$ arc $\left(\text{tang.} = \frac{\alpha}{x}\sqrt{\beta x + \gamma x^2}\right) - \frac{2\beta\alpha^3}{1+\gamma\alpha^2}\pi + \phi$ where $\phi = \frac{4\beta\alpha}{\sqrt{\gamma}\,(1+\gamma\alpha^2)}\,L\left(\frac{\sqrt{\gamma x}}{\sqrt{\beta}}\right) + \sqrt{\frac{\gamma x}{\beta} + 1}\Big)$, or $= \frac{4\beta\alpha}{\sqrt{-\gamma}\,(1+\gamma\alpha^2)}$ arc $\left(\text{sine} = \frac{\sqrt{-\gamma x}}{\sqrt{\beta}}\right)$ as ν or γ is positive or negative.

This is *the action of an infinitely long cylinder on a point at the vertex of its transverse section, the equation of the said section being* $y^2 = \alpha^2(\beta x + \gamma x^2)$.

Ex. If the base, or transverse section, is an ellipsis, or if $y^2 = \frac{b^2}{a^2}(ax - x^2)$, we have $\alpha^2 = \frac{b^2}{a^2}$, $\beta = a$, $\gamma = -1$; and $A = 4\left(x + \frac{ab^2}{a^2-b^2}\right)$ arc $\left(\text{tang.} = \frac{b}{ax}\sqrt{ax - x^2}\right) + \frac{4a^2b}{a^2-b^2}$ arc $\left(\text{sine} = \frac{\sqrt{x}}{\sqrt{a}}\right) - \frac{2ab^2}{a^2-b^2}\pi$. When $x = a$, this expression is reduced to

$$A = \frac{2ab}{a+b}\pi.$$

Scholium to Cor. 2, *Prop.* 30, page 281.

If we would have a general expression for the attraction of such solids as the one we considered in the proposition, when the guiding curve is any conic section, or when $y^2 = \alpha^2(\beta x + \gamma x^2)$, there arises at first (from the formula for the action of a rhombus)

$A = 4\int\dot{x}$ arc $\left(\text{tang.} = \frac{1}{rx}\sqrt{x^2 + (1+r^2)\,\alpha^2(\beta x + \gamma x^2)}\right) + 4\int\dot{x}$ arc $\left(\text{tang.} = \frac{1}{r'x}\sqrt{x^2 + (1+r'^2)\,\alpha^2(\beta x + \gamma x^2)}\right) - 2\pi x$, and by actually taking the fluent

$A = 4\left(x + \frac{\beta\alpha^2}{1+\gamma\alpha^2}\right)\left\{\text{arc}\left(\text{tang.} = \frac{1}{rx}\sqrt{\mu x + \nu x^2}\right) + \text{arc}\left(\text{tang.} = \frac{1}{r'x}\sqrt{\mu' x + \nu' x^2}\right)\right\} - \left(\frac{4\beta\alpha^3}{1+\gamma\alpha^2} + 2x\right)\pi + \phi + \phi'$,

Fig. 1.

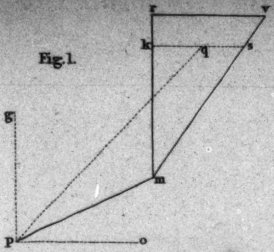


Fig. 2.

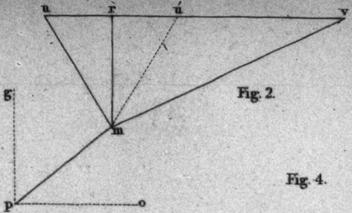


Fig. 3.

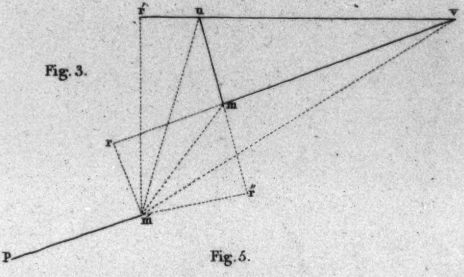


Fig. 4.

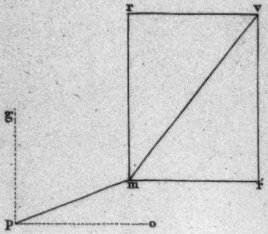


Fig. 5.

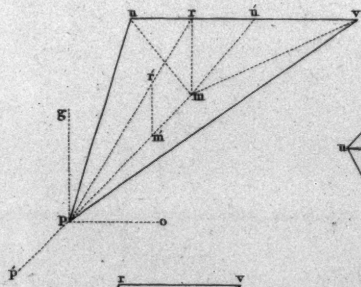


Fig. 6.

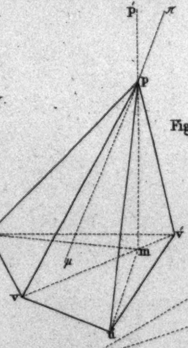
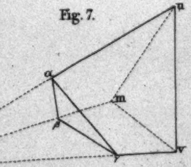


Fig. 7.



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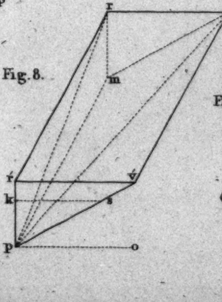


Fig. 9.

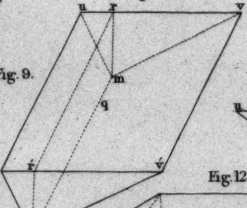


Fig. 10.

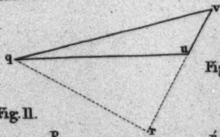


Fig. 11.

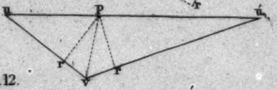


Fig. 12.

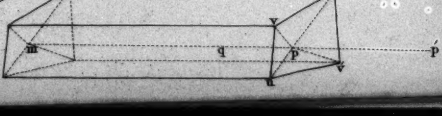


Fig. 13.

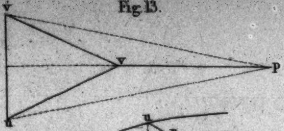
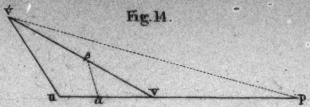


Fig. 14.



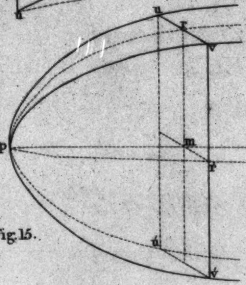


Fig. 15.

Fig. 16.

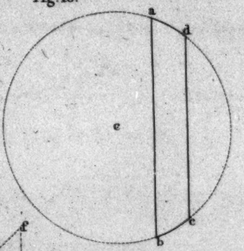


Fig. 17.

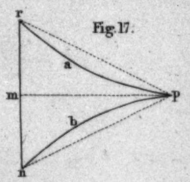


Fig. 18.

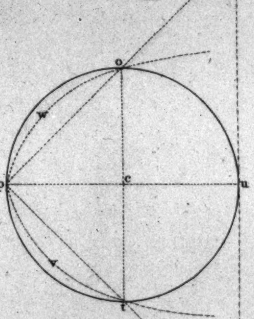


Fig. 19.

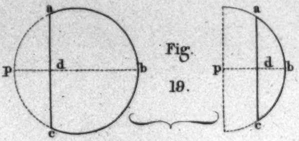
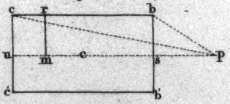


Fig. 20.



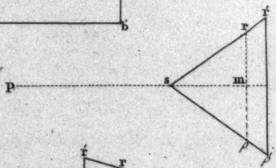


Fig. 21.

Fig. 22.

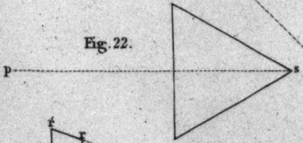
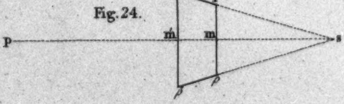


Fig. 24.



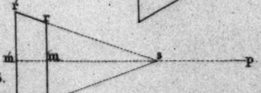


Fig. 23.

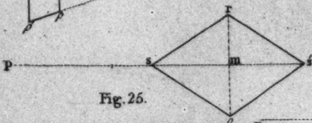
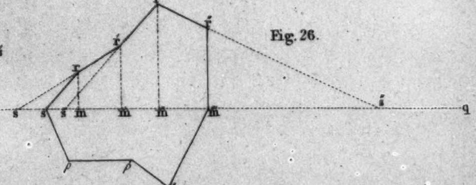


Fig. 25.

Fig. 26.



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where $\mu = \beta\alpha^2(1 + r^2)$, $\nu = 1 + \gamma\alpha^2(1 + r^2)$, $\mu' = \beta\alpha^2(1 + r'^2)$, $\nu' = 1 + \gamma\alpha^2(1 + r'^2)$,

$\varphi = \frac{4r\beta\alpha^3}{\sqrt{\nu}\,(1+\gamma\alpha^2)} \mathrm{L}\left(\frac{\sqrt{\nu x}}{\sqrt{\mu}} + \sqrt{\frac{\nu x}{\mu} + 1}\right)$, or $= \frac{4r\beta\alpha^3}{\sqrt{-\nu}\,(1+\gamma\alpha^2)}$ arc $\left(\text{sine} = \frac{\sqrt{-\nu x}}{\sqrt{\mu}}\right)$ as ν is positive or negative,

$\varphi' = \frac{4r'\beta\alpha^2}{\sqrt{\nu'}\,(1+\gamma\alpha^3)} \mathrm{L}\left(\frac{\sqrt{\nu' x}}{\sqrt{\mu'}} + \sqrt{\frac{\nu' x}{\mu'} + 1}\right)$, or $= \frac{4r'\beta\alpha^3}{\sqrt{-\nu'}\,(1+\gamma\alpha^2)}$ arc $\left(\text{sine}\ \frac{\sqrt{-\nu' x}}{\sqrt{\mu'}}\right)$ as ν' is positive or negative.

XV. *Of the Penetration of a Hemisphere by an indefinite Number of equal and similar Cylinders.* *By* Thomas Knight, *Esq.* *Communicated by Sir* Humphry Davy, *LL. D. Sec. R. S.*

Read March 19, 1812.

THE well known theorems of VIVIANI and BOSSUT, respecting certain portions of the surface and solidity of a hemisphere, form, together, a single case of the following problem; which is one of the most remarkable, for generality and simplicity of result, in the whole compass of geometry.

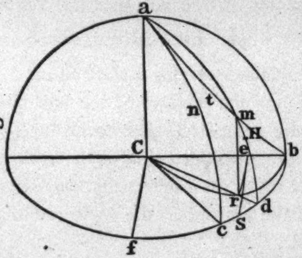

Fig. 1.

Problem.

To pierce a hemisphere, perpendicularly on the plane of its base, with any number of equal and similar cylinders; of such a kind, that, if we take away from the hemisphere those portions of the cylinders that are within it, the remaining part shall admit of an exact cubature: and if we take away, from the surface of the hemisphere, those portions cut out by the cylinders, the remaining surface shall admit of an exact quadrature.

Let fig. 1 represent the nearest half of the hemisphere, where a is the pole, bdcf a quadrant of the great circle forming its base. From every point d, on this side of b, draw the radius dC to the centre of the hemisphere, and (if the number of cylinders is to be $2n$*) take the arc bs equal to n times the arc bd, draw se perpendicular to Cb, and with the centre C and radius Ce describe the arc er cutting Cd in r. Through all the points (r) thus found, draw the curve line brC, terminated at b and C, and it shall be half the base of one of the required cylinders.

It is, in the first place, evident, from the construction, that the half cylinder, whose base is beCrb, is contained between two planes CabC, CacC, making with each other the angle bCc $= \frac{90°}{n}$; consequently the whole base of the hemisphere may be pierced by $2n$ such cylinders as this is the half of.

Let atmb be the intersection of the surfaces of the half cylinder and hemisphere; amd a great circle passing through a and d, and meeting atmb at m. Call the radius of the sphere r, Cr is the cosine of the arc bs to the radius r, by construction; it is also the cosine of the arc md to the same radius; therefore md $=$ bs $= n \times$ bd.

Put bd $= \varphi$; md $= n \times \varphi$; d$x = \psi$. Moreover, put A for the spherical space atmbdcna contained by the arcs anc, cdb and the curve atmb; and let S be the solidity of the portion of the hemisphere contained between the quadrant ancC and the surface (brCatmb) of the half cylinder. It is easy to see that

$$A = r^2 \iint \dot{\varphi} \cos. \psi \times \dot{\psi},$$

$$S = \frac{r^3}{2} \iint \dot{\varphi} \cos. \psi \times \cos. \psi \times \cos. \psi \, \dot{\psi}$$
$$- \frac{r^3}{2} \iint \dot{\varphi} \cos. n\varphi \times \cos. n\varphi \times \cos. \psi \, \dot{\psi}.$$

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* I do not intend $2n$ to represent an even number *only*, n may be $\frac{1}{2}$, or $\frac{3}{2}$, or $\frac{5}{2}$, &c. and $2n$ express any number whatever.

The fluents to be taken, first from $\psi = o$ to $\psi = n\varphi$, and then from $\varphi = o$, to $\varphi = \frac{90^\circ}{n}$. The first operation gives

$$A = r^2 \int \varphi \text{ sin. } n\varphi,$$

$$S = \frac{r^3}{2} \int \varphi \left\{ \frac{3}{4} \text{ sin. } n\varphi + \frac{1}{12} \text{ sin. } 3n\varphi \right\} - \frac{r^3}{2} \int \varphi \text{ sin. } n\varphi \text{ cos. }^2 n\varphi,$$

and by the second we get

$$A = C - \frac{r^2}{n} \text{ cos. } n\varphi$$

$$S = \frac{r^3}{2n} \left\{ \frac{\text{cos. }^3 n\varphi}{3} - \frac{3}{4} \text{ cos. } n\varphi - \frac{1}{36} \text{ cos. } 3n\varphi \right\} + C,$$

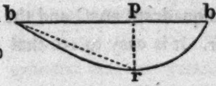
which fluents being taken from $n\varphi = o$, to $n\varphi = 90^\circ$, are $A = \frac{r^2}{n}$; $S = \frac{2}{9} \times \frac{r^3}{n}$; and if these are multiplied by $4n$, we have

$$A = 4r^2; \; S = \frac{8}{9} r^3;$$

for the whole that remains of the surface and solidity of the hemisphere after the subduction of the $2n$ cylinders. Thus A and S (for the whole hemisphere) do not depend on the number of the cylinders with which the penetration is made; *a most remarkable circumstance, seeing that amongst the bases of those cylinders are curves of an infinity of different kinds and orders.*

Let fig. 2 represent half the base of one of the cylinders; Cb the radius of the hemisphere, C the centre. From r, any point in the curve, let fall the perpendicular rp on the axis; call Cp, x; rp, y.

Fig. 2.



By construction, $Cr = \sqrt{x^2 + y^2} = r \text{ cos. } n \,.\, bCr$; now the cosine of the simple arc bCr is $\frac{x}{\sqrt{x^2+y^2}}$, which being put in the trigonometrical expression for the cosine of the multiple arc in terms of the cosine of the simple one, we have, for the equation of the curve brC.

When $n=1$, $\sqrt{x^2+y^2}=\frac{rx}{\sqrt{x^2+y^2}}$ or $x^2+y^2=rx$, the equation of a circle.

When $n=2$, $\sqrt{x^2+y^2}=\frac{2rx^2}{x^2+y^2}-r$ or $(x^2+y^2)^3=r^2(x^2-y^2)^2$; and in general the curve will be algebraic when n is any whole number.

XVI. *On the Motions of the Tendrils of Plants.* *By* Thomas Andrew Knight, *Esq. F. R. S. In a Letter to the Right Hon. Sir* Joseph Banks, *Bart. K. B. P. R. S.*

Read May 4, 1812.

My Dear Sir,

The motions of the tendrils of plants, and the efforts they apparently make to approach and attach themselves to contiguous objects, have been supposed by many naturalists to originate in some degrees of sensation and perception: and though other naturalists have rejected this hypothesis, few, or no experiments have been made by them to ascertain with what propriety the various motions of tendrils, of different kinds, can be attributed to peculiarity of organization, and the operation of external causes. I was consequently induced, during the last summer, to employ a considerable portion of time to watch the motions of the tendrils of different species of plants; and I have now the pleasure to address to you an account of the observations I was enabled to make.

The plants selected were the Virginia creeper (the ampelopsis quinquefolia of Michaux,) the ivy, and the common vine and pea.

A plant of the ampelopsis, which grew in a garden pot, was removed to a forcing house in the end of May, and a single shoot from it was made to grow perpendicularly upwards, by being supported in that position by a very slender bar of wood, to which it was bound. The plant was placed in the middle of the house, and was fully exposed to the sun; and every object around it was removed far beyond the reach of its tendrils. Thus circumstanced, its tendrils, as soon as they were nearly full grown, all pointed towards the north, or back wall, which was distant about eight feet: but not meeting with any thing in that direction, to which they could attach themselves, they declined gradually towards the ground, and ultimately attached themselves to the stem beneath, and the slender bar of wood.

A plant of the same species was placed at the east end of the house, near the glass, and was in some measure skreened from the perpendicular light; when its tendrils pointed towards the west, or centre of the house, as those under the preceding circumstances had pointed towards the north and back wall. This plant was removed to the west end of the house, and exposed to the evening sun, being skreened, as in the preceding case, from the perpendicular light; and its tendrils, within a few hours, changed their direction, and again pointed to the centre of the house, which was partially covered with vines. This plant was then removed to the centre of the house, and fully exposed to the perpendicular light, and to the sun; and a piece of dark-coloured paper was placed upon one side of it just within the reach of its tendrils; and to this substance they soon appeared to be strongly attracted. The paper was then placed upon the opposite side, under similar circumstances, and there it was soon followed by the tendrils. It was then removed, and a piece of plate glass was substituted; but to this substance the tendrils did not indicate any disposition to approach. The position of the glass was then changed, and care was taken to adjust its surface to the varying position of the sun, so that the light reflected might continue to strike the tendrils; which then receded from the glass, and appeared to be strongly repulsed by it.

The tendrils of the ampelopsis very closely resemble those of the vine, in their internal organization, and in originating from the alburnous substance of the plant; and in being, under certain circumstances, convertible into fruit stalks. The claws, or claspers, of the ivy, to experiments upon which I shall now proceed, appear to be cortical protrusions only; but to be capable (I have reason to believe) of becoming perfect roots, under favourable circumstances. Experiments, in every respect very nearly similar to the preceding, were made upon this plant; but I found it necessary to place the different substances, to which I proposed that the claws should attempt to attach themselves, almost in contact with the stems of the plants. I observed that the claws of this plant evaded the light, just as the tendrils of the ampelopsis had done; and that they sprang only from such parts of the stems as were fully, or partially, shaded.

A seedling plant of the peach tree, and one of the ampelopsis and ivy, were placed nearly in the centre of the house, and under similar circumstances; except that supports, formed of very slender bars of wood, about four inches high, were applied to the ampelopsis, and ivy. The peach tree continued to grow nearly perpendicularly, with a slight inclination towards the front and south side of the house, whilst the stems of the ampelopsis and ivy, as soon as they exceeded the height of their supports, inclined many points from the perpendicular line, in the opposite direction.

It appears therefore that not only the tendrils and claws of these creeping dependent plants, but that their stems also, are made to recede from light, and to press against the opake bodies, which nature intended to support and protect them.

M. **Decandole**, I believe, first observed that the succulent shoots of trees and herbaceous plants, which do not depend upon others for support, are bent towards the point from which they receive light, by the contraction of the cellular substance of their bark, upon that side, and I believe his opinion to be perfectly well founded. The operation of light upon the tendrils and stems of the ampelopsis and ivy appears to produce diametrically opposite effects, and to occasion an extension of the cellular bark, wherever that is exposed to its influence; and this circumstance affords, I think, a satisfactory explanation why these plants appear to seek and approach contiguous opake objects, just as they would do, if they were conscious of their own feebleness, and of power in the objects, to which they approach, to afford them support and protection.

The tendril of the vine, as I have already stated, is internally similar to that of the ampelopsis, though its external form, and mode of attaching itself, by twining round any slender body, are very different. Some young plants of this species, which had been raised in pots in the preceding year, and had been headed down to a single bud, were placed in a forcing-house, with the plants I have already mentioned; and the shoots from these were bound to slender bars of wood, and trained perpendicularly upwards. Their tendrils, like those of the ampelopsis, when first emitted, pointed upwards; but they gradually formed an increasing angle with the stems, and ultimately pointed perpendicularly downwards; no object having presented itself to which they could attach themselves.

Other plants of the vine, under similar circumstances were trained horizontally; when their tendrils gradually descended beneath their stems, with which they ultimately stood very nearly at right angles.

A third set of plants were trained almost perpendicularly downwards; but with an inclination of a few degrees towards the north; and the tendrils of these permanently retained very nearly their first position, relatively to their stems; whence it appears that these organs, like the tendrils of the ampelopsis, and the claws of the ivy, are to a great extent under the control of light.

A few other plants of the same species were trained in each of the preceding methods; but proper objects were placed, in different situations, near them, with which their tendrils might come into contact; and I was by these means afforded an opportunity of observing, with accuracy, the difference between the motions of these and those of the ampelopsis, under similar circumstances. The latter almost immediately receded from light, by whatever means that was made to operate upon them; and they did not subsequently shew any disposition to approach the points, from which they once receded. The tendrils of the vine, on the contrary, varied their positions in every period of the day, and after, returned again during the night to the situations they had occupied in the preceding morning; and they did not so immediately, or so regularly, bend towards the shade of contiguous objects. But as the tendrils of this plant, like those of the ampelopsis, spring alternately from each side of the stem, and as one point only in three is without a tendril, and as each tendril separates into two divisions they do not often fail to come into contact with any object within their reach; and the effects of contact upon the tendril are almost immediately visible. It is made to bend towards the body it touches, and, if that body be slender, to attach itself firmly by twining round it, in obedience to causes which I shall endeavour to point out.

The tendril of the vine, in its internal organization, is apparently similar to the young succulent shoot, and leaf-stalk, of the same plant; and it is as abundantly provided with vessels, or passages, for the sap; and I have proved that it is alike capable of feeding a succulent shoot, or a leaf, when grafted upon it. It appears therefore, I conceive, not improbable, that a considerable quantity of the moving fluid of the plant passes through its tendrils; and that there is a close connection between its vascular structure and its motions.

I have proved in the Philosophical Transactions of 1806, that centrifugal force, by operating upon the elongating plumules of germinating seeds, occasions an increased growth and extension upon the external sides of the young stems, and that gravitation produces correspondent effects; probably by occasioning the presence of a larger portion of the fluid organizable matter of the plant upon the one side, than upon the other. The external pressure of any body upon one side of a tendril will probably drive this fluid from one side of the tendril, which will consequently contract, to the opposite side, which will expand; and the tendril will thence be compelled to bend round a slender bar of wood or metal, just as the stems of germinating seeds are made to bend upwards, and to raise the cotyledons out of the ground; and in support of this conclusion I shall observe, that the sides of the tendrils, where in contact with the substance they embraced, were compressed and flattened.

The actions of the tendrils of the pea were so perfectly similar to those of the vine, when they came into contact with any body, that I need not trouble you with the observations I made upon that plant. An increased extension of the cellular substance of the bark upon one side of the tendrils, and a correspondent contraction upon the opposite side, occasioned by the operation of light, or the partial pressure of a body in contact, appeared in every case, which has come under my observation, the obvious cause of the motions of tendrils; and therefore, in conformity with the conclusions I drew in my last memoir, respecting the growth of roots, I shall venture to infer, that they are the result of pure necessity only, uninfluenced by any degrees of sensation, or intellectual powers.

I am, my dear Sir,

with much regard, &c.

THO. ANDREW KNIGHT.

Downton, April 27, 1812.

To the Right Hon. Sir Joseph Banks, Bart. K. B.

XVII. *Observations on the Measurement of three Degrees of the Meridian conducted in England by Lieut. Col.* William Mudge. *By Don* Joseph Rodriguez. *Communicated by* Joseph de Mendoza Rios, *Esq. F. R. S.*

Read June 4, 1812.

The determination of the figure and magnitude of the earth has at all times excited the curiosity of mankind, and the history of the several attempts made by astronomers to solve this problem might be traced to the most remote antiquity. But the details of the methods pursued by the ancients on this subject being extremely vague, and their results expressed in measures of which we do not know the relation to our own, in fact give us very little assistance in learning either the figure or dimensions of our globe.

It was not till the revival of science in Europe that the two great philosophers, Huyghens and Newton, first engaged in the consideration of this question, and reduced to the known laws of mechanics, the principles on which the figure of the earth should be determined.

They demonstrated that the rotatory motion should occasion differences in the force of gravity in different latitudes, and consequently that parts of the earth in the neghbourhood of the equator should be more elevated than those near the poles.

The most simple hypothesis, which first presented itself to their imagination, was that which supposed the earth to be throughout composed of the same kind of matter, and its surface that of a spheroid generated by revolution round its axis. This hypothesis, adopted by NEWTON only as an approximation to the truth, is, in fact, perfectly consistent with the equilibrium to which particles in a state of paste, or of tardy fluidity, would arrive in a short time after their present motion was impressed; and the eccentricity derived from this hypothesis is at least not very remote from that which actually obtains in the present state of consistence and stability which the earth has since acquired.

But the homogeneity of the matter, of which the earth consists, is at variance with all geological observations, which prove evidently that at least 5000 toises of the exterior crust is formed of an immense mass of heterogeneous matters varying in density from each other; and upon the supposition of a state of fluidity of the whole, it should follow that the strata should successively increase in density from the surface towards the centre, that the more dense would accordingly be subjected to less of centrifugal force, and consequently that the spheroidical form resulting from this cause would be less eccentric than would arise from a state of perfect homogeneity.

The most simple, as well as the most effectual means of verifying the hypothesis respecting the figure of the earth, is to measure in the two hemispheres several arcs of its meridians in different latitudes, at some distance from each other. On this subject it must be allowed, that the Academy of Sciences at Paris set the example, in giving the original impulse to the undertaking, and not only commenced, but put in execution those parts of the plan which were most difficult and most decisive.

The results of the first measurements made of different arcs on the meridian of different parts of the world, were found to be perfectly conformable to the expectations of HUYGHENS and of NEWTON, and also with experiments made on the vibration of the pendulum in different latitudes; and they left no doubt that the earth was in fact flattened at the poles; establishing thereby one point extremely interesting in natural philosophy.

These results, however, did not correspond with sufficient accuracy for ascertaining with precision the degree of eccentricity, or even the general dimensions of the earth, as might naturally be expected when we consider the necessary imperfection of the means then employed in these operations, and the great difficulties that are to be encountered.

For the purpose of making a nearer approximation to the true dimensions of the earth, and of verifying former measurements, it is necessary in some instances to repeat them, and also to make others in different situations, which may be expected to be improved in proportion to the progress that is made in the means of perfecting the several departments of science.

At the commencement of the French revolution, men of science took advantage of the general impulse which the human mind received in favour of every species of innovation, or change, and they proposed making a new measurement of an arc of the meridian in France, for the purpose of establishing a new system of weights and measures, which should be permanent, as being founded on the nature of things.

A commission, composed of some of the most distinguished members of the Academy of Sciences, was charged to form the plan of these operations, which were to serve as the basis of the new system. They invented new instruments, new methods, new formulæ, and in short almost the whole of this important undertaking consisted of something new in science.

Two celebrated astronomers, Delambre and Mechain, were engaged to perform the astronomical and geodetical observations, and these they continued as far as Barcelona in Spain. The details of their operations, observations, and calculations, were subsequently examined by a committee of men of science, many of whom were foreigners collected at Paris, who confirmed their results, and by the sanction of such an union of talents, gave such a degree of credit and authenticity to their conclusions as could scarcely be acquired by other means.

Since that time, in the year 1806, Messrs. Biot and Arago, members of the National Institute, were sent into Spain for the express purpose of carrying on the same course of operations still further southward, from Barcelona as far as Formentera, the southernmost of the Balearic islands. Fortunately this last undertaking, which forms a most satisfactory supplement to the former, was completed by the month of May, 1808, at a period when political circumstances would not admit of any further operations being pursued, as a means of verifying the results, by measuring a base which should be independent of those formerly obtained in France.

In the year 1801, the Swedish Academy of Sciences, encouraged by the success of the operations conducted in France, sent also three of its members into Lapland, to verify their former measurement taken in 1736, by new methods, and by the use of new instruments, similar to those which had recently been used in France, and of which the National Institute made a handsome present to the Swedish Academy. The results of this new undertaking, which terminated in 1803, were drawn up by M. Svanberg, and are highly interesting, by their exactness, by the perspicuity of the details, and even a certain degree of novelty given to the subject by the arrangement adopted by the learned author M. Svanberg.

These new measures were found to confirm, in a remarkable manner, the general results of those which had preceded, and gave very nearly the same proportion for the eccentricity and other dimensions of the globe, so that there would not have remained the smallest doubt respecting the figure of the earth being flattened at the poles, had there not been a fourth measurement performed in England at the same time as that undertaken in Lapland, the results of which were entirely the reverse. This measurement, which comprised an arc of 2° 50′, was undertaken by Lieut. Col. Mudge, Fellow of the Royal Society, with instruments of the most perfect construction that had ever yet been finished by any artist, contrived and executed for that express purpose, by the celebrated Ramsden. The details of the observations and other operations of Lieut. Col. Mudge, may be seen in the volume of the Philosophical Transactions for the year 1803; and one cannot but admire the beauty and perfection of the instruments employed by that skilful observer, as well as the scrupulous care bestowed on every part of the service in which he was engaged. Bengal lights were employed on this occasion, as objects at the several stations, and their position appears to have been determined with the utmost precision by the theodolite of Ramsden, which reduces all angles to the plane of the horizon, and with such a degree of correctness, that the error in the sum of the three angles of any triangle is scarcely, in any instance, found to exceed three seconds of a degree, and in general not more than a small fraction of a second.

Accordingly the geodetical observations were conducted with a degree of exactness, which hardly can be exceeded; and even if we suppose for a moment, that the chains made use of in the measurement of the bases may not admit of equal precision with the rods of platina employed in France, nevertheless, the degree of care employed in their construction, in the mode of using them, and the pains taken to verify their measures was such, that no error that can have occurred in the length of the base, could make any perceptible difference in the sides of the series of triangles, of which the whole extent does not amount to so much as three degrees.

Nevertheless, the results deduced by the author, from this measure alone, would lead to the supposition that the earth, instead of being flattened at the poles, is in fact more elevated at that part than at the equator, or at least, that its surface is not that of a regular solid. For the measures of different degrees on the meridian, as reduced by Lieut. Col. Mudge, increase progressively toward the equator.

The following table of the different measures of a degree in fathoms is given by the author in his Memoir.

Latitude.	
52° 50′ 30″	60766
52 38 56	60769
52 28 6	60794
52 2 20	60820

Latitude.	
51° 51′ 4″	60849
51 25 18	60864
51 13 18	60890
51 2 54	60884

The singularity of these results excites a suspicion of some incorrectness in the observations themselves, or in the method of calculating from them. The author has not informed us in his Memoir, what were the formulæ which he employed in the computations of the meridian; but one sees, by the arrangement of his materials, that he made use of the method of the perpendiculars without regard to the convergence of the meridians; and although this method is not rigorously exact, it can make but a very few fathoms more in the total arc, and will have very little effect on the magnitude of each degree. It is therefore a more probable supposition, that, if any errors exist, they have occurred in the astronomical observations. But it is scarcely possible to determine the amount of the errors, or in what part of the arc they may have occurred, excepting by direct and rigorous computation of the geodetical measurement. I have therefore been obliged to have recourse to calculations, which I have conducted according to the method and formulæ invented and published by M. Delambre.

The means generally employed for finding the extent of a degree of the meridian, consists in dividing the length of the total arc in fathoms, by the number of degrees and parts of a degree deduced from observations of the stars; but if these observations are affected by any error, arising from unsteadiness of the instrument, from partial attractions, or from any other accidental causes, then the degrees of the meridian will be affected, without a possibility of discovering such an error in this mode of operating. It is consequently necessary, in such a case, to employ some other method, which may serve as a means of verifying the observations themselves, of detecting their errors, if there be any, or at least of shewing their probable limits.

My object therefore is to communicate the result of calculations that I have made, from the data published by Lieut. Col. MUDGE in the Philosophical Transactions; and I hope to make it appear, that the magnitude of a degree of the meridian, corresponding to the mean latitude of the arc measured by this skilful observer, corresponds very exactly with the results of those other measurements that have been above noticed.

In M. DELAMBRE'S method nothing is wanting but the spherical angles, that is to say, the horizontal angles observed, corrected for spherical error. Moreover, for our purpose, we have no occasion for the numerical value of the sides of the series of triangles, but only for their logarithms. Thus the logarithm of the base measured at Clifton, as an arc gives us that of its sine in feet or in fathoms, so that by means of this latter logarithm, and the spherical angles of the series of triangles, we obtain at once, and as easily as in plane trigonometry, the logarithms of the sines of all their sides in fathoms.

After this, it is extremely easy to convert them into logarithms of chords or of arcs, for the purpose of applying them to the computation of the arcs on the meridian or azimuths. I give the preference to taking the logarithms of the sides as arcs, because the computations become in that case much more simple and expeditious.

Near to Clifton, which is the northern extremity of the arc, in a situation elevated 35 feet above the level of the sea, a base was measured of 26342,7 feet in length, the chains being supposed at the temperature of 62° FAHRENHEIT, or $13\frac{1}{3}$° REAUMUR.

For reducing this base to toises, we have the proportion of the English foot to that of France, as 4 : 4,263, so that if p be taken to express the fractional part of the French foot, corresponding to English measure, then log. $p = 9{,}97234{,}46587$, and then log. of $26{,}342{,}7 = 4{,}42066{,}02860$, and hence the log. of the base in toises will be found equal to 3,61485,36943, and the number of toises corresponding is 4119,5 taken at the same temperature, which corresponds to $16\frac{2}{3}$° of the centigrade thermometer.

This base we must consider as an arc of a circle, and it is easy to reduce it to the sine of the same arc, according to the method given in a note at the end of this memoir. The logarithm of the *sine* of the base in toises is found to be 3,61485,35800.

With this quantity as base, and by means of the spherical triangles given by Lieut. Col. MUDGE in his paper, I have found the logarithmic sines in toises of all the sides of his series of triangles, and have subsequently reduced them to logarithmic arcs of the same, which enable me to complete the rest of the calculation. With these we may compute any portions of the meridian, or successive intervals of different stations expressed in toises, and in parts of the circle, or their respective azimuths, having regard always to the relative convergence of different meridians.

The author has made observations for determining the latitude of the two extremities of his arc, and has also determined the azimuths of the exterior sides in his series of triangles by means of the greatest elongation of the pole star.

In the calculations that I have made, I began at Clifton in Yorkshire, the northern extremity of the arc, and for this purpose the following are the data furnished by Lieut. Col. Mudge.

Latitude of Clifton reduced to the centre of the station 53° 27′ 36,″62.

Azimuth of Gringley, seen from Clifton, and reckoned from the north toward the west 256° 17′ 25″.

Azimuth of Heathersedge, seen from Clifton, and reckoned in the same direction 118° 8′ 8″,81.

With these data, and the two tables of spherical triangles, and the logarithms of their sides expressed in arcs, the intervals between Clifton and the two stations Gringley and Heathersedge were found in toises and in seconds of a degree, as well as all the corrections to be made on the first azimuths increased by 180°, as azimuths of Clifton seen on the horizon at these latter places.

The same process was continued for the following stations in succession, all the way to Dunnose in the Isle of Wight, which is the southernmost extremity of the series.

In this manner we have the latitudes and azimuths of each station, by means of two or three preceding stations, and consequently we have a verification of all the calculations that have been before made by Lieut. Col. Mudge.

The results of my calculations are contained in the two following tables.

First Table of Distances in Toises and in Seconds of a Degree on the Meridian, comprised between the westerly Stations in the Series of Triangles.

Names of the Stations.	Arcs in Toises.	Arcs in Seconds.
Clifton -	0,0	0,0
Heathersedge	6834,324	480,9928
Orpit -	15818,489	997,5928
Castlering -	19801,1934	1248,8226
Corley -	14295,384	901,6207
Epwell -	22327,008	1408,2543
Stow -	9555,479	602,7284
Whitehorse	18799,645	1185,8656
Highclere -	14990,567	945,6354
Dean Hill -	16105,614	1016,0180
Dunnose -	23529,886	1484,4531
Sum total -	162057,5437	10221,9837

Second Table of successive Intervals between the Eastern Stations.

Names of the Stations.	Arcs in Toises.	Arcs in Seconds.
Clifton -	0,0	0,0
Gringley -	2809,105	177,149
Sutton -	10838,816	1061,931
Holland Hill	4681,190	295,2251
Bardon Hill -	18092,261	1141,0462
Arbury Hill	27956,417	1763,2683
Brill -	22374,106	1411,2769
Nuffield -	14350,3834	905,2155
Bagshot -	12137,933	765,6822
Hindhead -	14449,2027	911,5140
Butser Hill -	7853,644	495,4551
Dunnose -	20514,036	1294,1974
Sum total -	162057,0941	10221,9607

Now if we take the arithmetic mean of the sums contained in the two tables, we have for measures of the entire arc, comprised between the stations of Clifton and Dunnose, the following quantities 162057,32 toises, and 10221,972 seconds of a degree, or 2° 50′ 21″,972. By dividing the former of these by the second, we get the measure of a degree, corresponding to the mean latitude of the whole arc, equal to 57073,74 toises, or 60826,34 fathoms, at the temperature of $16\frac{2}{3}°$ of the centigrade thermometer, the latitude being 52° 2′ 20″.

The station at Arbury Hill happens to be very nearly in the meridian of Clifton and Dunnose, and divides the interval between them into nearly equal parts. The measures of that part of the arc, which lies between Arbury and Dunnose, is by the tables 91679,47 toises, and 9783″,34 seconds, or 1° 36′ 23″,34 of the common division of the circle. The mean latitude of the arc is 51° 25′ 21″. And the measure of 1 degree corresponding to it is 57068,41 toises.

In the same manner the measure of the arc comprised between Arbury Hill and the northern extremity at Clifton, is 70377,85 toises, and 4438,63 seconds, or 1° 13′ 58″,63. Its mean latitude is 52° 50′ 32″. And we have for one degree of the meridian, corresponding to this latitude, 57080,70 toises.

Hence, if we divide the entire arc into two equal parts, we deduce the following values of a degree corresponding to the middle of the whole and of its parts.

Latitudes.	
51° 25′ 20″	57068
52 2 20	57074
52 50 30	57081

These values are, as appears, perfectly in conformity with the theory, and with the results of other measures that have been taken in different parts of the northern hemisphere; but, in order to place that agreement in a more distinct point of view, I shall show how nearly these estimates agree with the elliptic hypothesis, by comparing them with those measures of a degree, on which we can place the greatest reliance for exactness.

Now, if we compare the results of these calculations with those deduced by Lieut. Col. Mudge from his observations, we shall see the probable source of those *errors*, which it appears to me have led him to false conclusions. It has already been observed, that the station at Arbury Hill divides the whole arc into two parts nearly equal, and that it is also nearly in the meridian of the two extremities at Dunnose and Clifton. It was, in all probability, this circumstance which determined the author to observe the latitude of Arbury Hill, as he would then have two partial arcs independent of the whole and of each other.

For determining the angular extent of these arcs, Lieut. Col. Mudge observed the zenith distances of several stars on the meridian above the pole, by means of a large zenith sector constructed by Ramsden, with the same pains that he had bestowed upon the theodolite. Lieut. Col. Mudge paid all possible attention, and took all such precautions as might naturally be expected from an observer of his experience and address. Nevertheless the results of his observations made on different stars, differ no less than 4 seconds from each other. But, by taking a mean of all, the dimensions of the three arcs reduced to the centre at each station are as follows.

Between Clifton and Dunnose	2° 50′ 23″,35
Clifton and Arbury	1 14 3 ,40
Arbury and Dunnose	1 36 19 ,95

The extent of the first arc, in linear measure, is 1036339½ feet English, and when this is reduced to toises, we have for the lengths of the three arcs from Lieut. Col. MUDGE's measures,

From Clifton to Dunnose	162067,3
Clifton to Arbury	70380,2
Arbury to Dunnose	91687,1

These last values exceed those resulting from my computations, the first by 10 toises, the second by 2, the third by 8 toises; and these differences arise from the convergence of the meridians, which the author thought might safely be neglected, and in fact it does not make a difference that is perceptible in the value of a degree upon the meridian. For the difference of 8 toises, in the distance between Dunnose and Arbury, makes but 5 toises difference in the value of a degree upon that arc, and the difference of 10 in the whole distance from Dunnose to Clifton, makes 3½ in the measure of each degree on that arc. So that, as far as this source of disagreement is concerned, the author's results and mine would not be found to differ materially from each other.

But, if we attend to the angular dimensions of the several arcs, as deduced from observation and from calculation, these will not be found to agree so nearly.

The following table will shew the differences in each instance.

Clifton and Dunnose	2° 50′ 23″,35	observed
	2 50 21 ,97	calculated
Difference	+ 1 ,38	
Clifton and Arbury	1° 14′ 8″,40	observed
	1 13 58 ,63	calculated
Difference	+ 4 ,77	
Arbury and Dunnose	1° 36′ 19″,95	observed
	1 36 23 ,34	calculated
Difference	− 3 ,39	

These differences are really considerable, and are capable of producing important errors in the results dependent on them.

In the first place we see, that the southernmost arc between Dunnose and Arbury is smaller than it would appear by computation, by as much as 3″,4, and when this deficiency is combined with an excess of 8 toises in the linear dimensions of the same arc, it makes as much as 40 toises difference in the estimated length of a degree. The reverse of this occurs in the northern portion of the arc comprised between Clifton and Arbury Hill. This is larger than it ought to be by 4″,77, and hence the value of a degree on the meridian turns out too small by about 62 toises in its linear dimensions. Fortunately, however, the excess of the total arc is extremely small, as it does not exceed 1″,38, so as to make but 5 or 6 toises difference in the length of a degree observed on the meridian, and corresponding to the mean latitude of the arc examined.

From what has been above stated, it seems almost beyond a doubt that it is to errors in the observations of latitude, that the appearance of progressive augmentation of degrees towards the equator, as represented by Lieut. Col. MUDGE in his paper, are to be ascribed, and that it is especially at the intermediate station at Arbury Hill, that the observations of the stars are erroneous nearly 5 seconds, notwithstanding the goodness of the instruments, and the skill and care of the observer. But, before I insist farther on this head, I will answer one objection that may be made to the principles of the method that I have pursued in this Memoir.

Those astronomers, who have hitherto undertaken the measurement of degrees of the meridian, have deduced their measures by simply dividing the linear extent by the number of degrees and minutes found by observation of the fixed stars taken at the two extremities of the arc. This is indeed the most simple that can be adopted; and it has the advantage of being independent of the elliptic figure of the earth, especially in arcs of small extent. The elements dependent on this figure, are too uncertain to be employed in calculating the angular intervals in the short distances between successive stations, even as a means of verification, without risk of committing greater errors than those to which astronomical observations can be liable. Accordingly one cannot safely make any use of it in cases where great accuracy is required.

I must admit the justness of this objection, and must therefore shew the extent to which it really applies to the present subject.

In the first place, I may suppose, that in consequence of some fault in the instrument, with respect to vertical position, construction, or some accidental derangement, there is an error of some seconds in the observations of the fixed stars. How is this to be discovered? This is not to be done by comparing the value of a degree on the meridian, as deduced from these observations, with the results of other measurements in distant parts of the globe. For if we find that these degrees so taken do not agree in giving the same ellipsoid, we are not to attribute all the differences to irregularities of the earth, without supposing any error on the part of the observer, of his instrument, or of other means employed in his survey.

But this, in fact, is what has generally been done. It must, however, be acknowledged, that the majority of observers have not been in fault, as they could do nothing better; but too much reliance has been placed on the goodness of their instruments, their means, and other circumstances. It is true that irregularities of the earth and local attractions may occasion considerable discrepancies which are even inevitable; but before we decide that these are the real source of disagreement, we ought carefully to ascertain that there are no others.

But to return to our subject, of the English measurement. If the uncertainty which yet subsists, with respect to the exact figure of the earth and its dimensions, occasions some small errors in the calculation of the series of triangles, the sum of these errors will be found in the estimate of the entire arc, and will increase in proportion to the extent of the arc measured. Now, in the English measurement, we find exactly the reverse of this. For the difference between the results of calculation and observation is only 1″,38 on the whole arc; but is even as high as 4″,77 on one of the smaller arcs. So that, whatever error we may suppose to have been introduced into the calculation by assuming a false estimate of the spheroidity of the earth, or of other elements employed in the calculation, it is very evident that the zenith distances of stars taken at Arbury Hill are affected by some considerable error, wholly independent of these elements.

It was not till the date of the measurement of the meridian in France, that M. Delambre published and explained, with admirable perspicuity and elegance, all the formulæ and methods relative to the calculation of spheroids, and put it in the power of astronomers in general to make use of the elliptic elements in verifying the results of their observations. In the present state of science these elements are well known, and the errors that can arise from any uncertainty in them, are not so considerable as is generally supposed. The oblateness and the diameter at the equator are the only elements wanting in the calculation; for the purpose of seeing what effect our present uncertainty respecting them can have on the subject in question, I have employed three different estimates of the oblateness $\frac{1}{330}$, $\frac{1}{320}$, and $\frac{1}{310}$. With respect to the radius of the equator, that is ascertained with sufficient precision by the mean of the arc extending from Greenwich to Formentera, corresponding to latitude 45° 4′ 18″. The value of the degree in toises is 57010,5, and it is highly probable that in this estimate the error does not amount to so much as half a toise, as it is deduced from an entire arc of 12° 48′ between the two extremities, the latitudes of which have been determined with extreme care, and by a great number of observations.

The following are the logarithms of radius at the equator, which I have employed as adapted to each degree of oblateness, and opposite to them are placed the corresponding computed estimate of the entire arc between Clifton and Dunnose.

$\frac{1}{330}$ . . . . 6,5147,400 . . . . . 2° 50′ 21,972
$\frac{1}{320}$ . . . . 6,5147,485 . . . . . 2° 50′ 21,974
$\frac{1}{310}$ . . . . 6,5147,670 . . . . . 2° 50′ 21,592

so that the greatest difference is but 0″,38. Let us suppose it 0″,4, or even 0″,5, for the second calculation was made only by means of the western series of triangles, and the third only with the eastern; but even then the error arising from uncertainty in the elements is not half the difference we find between the results of computation and of observations of the fixed stars. It appears therefore, that these elements are by no means to be neglected as a method of verification; and in fact the quantity of 1″,38 is so small, that it is extremely difficult to ascertain this quantity with the very best instruments. Of this we shall find further proof hereafter; but as this discussion is not without its use, I shall enter into some details on this subject.

The measurement in Lapland was performed by means of a double metre, and with a repeating circle of Borda, sent by the National Institute of France. In order to see to what degree of accuracy the arc computed would agree with that obtained by observations of the pole star above and below the pole, I assumed an oblateness of $\frac{1}{320}$, and as logarithm of radius I had 6,5147500 expressed in toises and in round numbers. With these elements, and with the data to be found in the work of M. Svanberg, we have by the western series of triangles 5840″,196 and 5840″,138 by the eastern. So that the mean calculated arc is 1° 37′ 20″,167, while the arc observed was 1° 37′ 19″,566. The difference then is 0″,6 for the total arc, and 0″,37 for the mean degree, or 5,86 toises excess in the linear extent. One can never depend upon quantities so small as this, so that the agreement between the results of computation and actual observation, proves not only the skill of the observers and the accuracy of which their instruments admit; but also that the elliptic elements employed in the calculation are a sufficiently near approximation to the truth to be deserving of confidence.

In the 8th volume of the Asiatic Researches, published by the Society at Calcutta, are contained the details of another measurement performed in 1802, by Major William Lambton in Bengal, on the Coromandel coast. In this undertaking, which was executed with great skill and attention, Major Lambton employed Bengal lights as signals, chains for the linear measures, and a theodolite, and a zenith-sector made by Ramsden. The base measured was 6667,740 fathoms reduced to the level of the sea, and to the temperature of 62° Fahrenheit; and the stations were so chosen, that four of the sides of the triangles were almost in the same line, and nearly parallel to the meridian at the southern extremity of the arc, so that their sum but little exceeds its whole extent. The lengths of these arcs in fathoms reduced to the meridian are thus given in the Memoir of Major Lambton.

AB 20758,13 north latitude of A 11° 44′ 52″,59
BC 17481,245
CD 22237,04 north latitude of E 13° 19′ 49″,018
DE 35246,43

From these data Major Lambton deduces the degree of the meridian to be 60485 fathoms, or 56762,3 toises. By applying to this the same elements as we did to the measurement by **Svanberg**, we have the entire arc measured equal to 1° 34′ 55″,896; so that the difference between the results of calculation and of the observations, is only 0″,532 for the whole arc, or 0″,337 for the mean degree. The elliptic hypothesis and observation agree more correctly in this instance, for the difference is rather less than in that of Lapland, although the two arcs are very nearly of the same extent. Thus the degree on the meridian measured in Bengal, in the latitude of 12° 32′ 21″ north, cannot be supposed to exceed Major **Lambton**'s estimate by more than 5,22 toises; and it is extremely difficult to speak with certainty to quantities so small as this.

The same observer also measured one degree perpendicular to the meridian, by means of a large side of one of his triangles cutting the meridian nearly at right angles, and of which he observed the azimuth at the two extremities. The data from which his results may be verified are these:

Length of the chord of the long side in English feet AB = 291197,20.

Azimuth of the eastern extremity A equal to 87° 0′ 7″,54 NW.

Azimuth of the western extremity B equal to 267° 10′ 44″,07 NW.

North latitude of A 12° 32′ 12″,27
North latitude of B 12° 34′ 38″,86.

With these data in the triangle formed by the long side, the meridian at B, and the perpendicular from B on the meridian at A, we have the chord of this last arc equal to 290845,8 feet, and the arc itself 290848,03 feet. By applying the method of M. **Delambre**, we find the azimuth of the extremity B less by 2″ than it was observed to be; so that we have no reason to suppose a greater error than one second in the observation of each azimuth, and it seems next to impossible to arrive at greater exactness.

The difference of longitude between the points A and B is 48′ 57″,36. With this angle and the co-latitude at A, we have in the spherical triangle right angled at the point A, the extent of the normal arc equal to 2867,330 seconds, and dividing its length in feet by this number, we have for the degree perpendicular to the meridian, at the extremity A, 60861,20 fathoms, or 57106,5 toises. Now these values are precisely what we find on the elliptic hypothesis, with an oblateness of $\frac{1}{320}$ or $\frac{1}{310}$; and in short, the correspondence between the hypothesis and the measures of Major Lambton, is as complete as can be wished. Major Lambton, indeed, finds the degree on the perpendicular too great by 200 fathoms, but this arises from a mistake in his calculation.

Lastly, I shall apply the same method, and see how nearly the elliptic hypothesis agrees with the last measures taken in France, which merit the highest degree of confidence both with respect to the observers who have executed it, and the means which they had it in their power to employ. I have taken only the arc between Dunkirk and the Pantheon at Paris, from the data published by the Chevalier Delambre in the 3d Vol. of the Measurement of the Meridian. I employed the same elements and similar calculations to those made on the English arc. The oblateness of $\frac{1}{330}$ gives the difference between the parallels equal to 7883,615 seconds by the eastern series of triangles, and 7883,617 by the western series. The mean of these 7883,616 may be taken as the true extent of the total arc.

The two other elements give for this quantity 7883″,621 and 7883″,493, or 2° 11′ 23″,6 and 23″,49, as the calculated extent of the arc. But the arc observed was 2° 11′ 19″,83, according to M. Delambre, and 2° 11′ 20″,85 according to M. Mechain; so that the least difference between the calculation and the observations will be 2″,64. M. Delambre is of opinion, that the latitude of Dunkirk, which is supposed to be 51° 2′ 9″,20, should be diminished; and in fact the distance between the parallels of Dunkirk and Greenwich, which is 25241,9 toises, gives by the mean of the three assumed ellipticities 26′ 32″,3 for the difference of latitude. After deducting this quantity from 51° 28′ 40″, the supposed latitude of Greenwich, there remains 51° 2′ 7″,7 or 8″, for that of the tower at Dunkirk. If from this again we deduct the calculated arc 2° 11′ 23″,5, we have 48° 50′ 44″,5 for the latitude of the Pantheon, while, according to the observations of M. Delambre, it is 49″,37, or 48″,35 by those of M. Mechain. If various circumstances, with regard to unfavourable weather, and also others of a different kind connected with the revolution, and of which M. Delambre complains with much reason, have occasioned some uncertainty with respect to the observations at Dunkirk, still the numerous observations made at Paris, both by him and by M. Mechain at a more favourable season, and in times of perfect tranquillity, render the supposition of an error of 4 seconds in the latitude of the Pantheon wholly inadmissible. It is, however, too true, that such errors are possible, and it is only by careful perseverance, and by repeated verification, that they are to be discovered and removed, as we have seen to be highly probable with respect to the station at Arbury Hill.

But the same celebrated observer, M. Mechain, who handled instruments with great delicacy, and was possessed of peculiar talents for this species of observation, has given us an instance of singular irregularity in the observations made at Montjui and at Barcelona.

The latitude of Montjui, determined by a very long and regular series of zenith distances, is full 3″,24 less than that deduced from a similar series of observations made at Barcelona, with the very same instruments, and with equal care. Moreover, there is reason to think, from other observations, that the latitude of Barcelona (which is supposed to be 45″) ought to be diminished still one second, so that the difference between the observations at Montjui and at Barcelona will probably amount to as much as 4″. Local attractions are supposed to have been the cause of this irregularity; but then the latitude, as deduced from observations made at Barcelona, should have been less than it appeared by those made at Montjui itself; for the deviation of the plumb-line (or of the spirit contained in a level) *could only* be occasioned by the little chain of land elevated to 120 or 130 toises, which passes to the north of Barcelona in a north-easterly direction. Now since the deviations arising from this source would be northward, the zenith distance of circumpolar stars would be augmented by that deviation, and consequently the latitude deduced therefrom would be diminished by just so much. But here the contrary occurs; for the latitude of Montjui deduced from the observations at Barcelona is 48″,23, whilst that obtained by direct observations at Montjui is only 45″. Hence it seems probable, that the cause of this irregularity must be sought elsewhere, and that it is not likely to be discovered without repeating over again the same observations.

Moreover it does not follow that the latitudes of two places are correct, because the declinations of the stars deduced from them correspond; for the deviations caused by local attractions, or from any other source, are made to disappear in correcting the declination, but remain uncorrected in the latitude of each.

Lieut.Col. **Mudge** is also of opinion, that the irregularity in the value of his degree may be ascribed to deviation of the plumb-line, occasioned by local attractions. This is certainly very possible, and may be decided by an examination of all circumstances on the spot. But if there be really an error of 1″ in the extent of the whole arc, this should rather be ascribed to some defect in the observations themselves, than to any extraneous source; for the observations of different stars give results that differ more than 4 seconds from each other.

I shall now conclude this Memoir, by expressing a wish, which men of science in England have it more in their power than any others to gratify; I mean by making new measurements in the southern hemisphere. Those which have been made hitherto in the northern hemisphere are extremely satisfactory by their agreement, and give us great reason to presume that the general level of the earth's surface is elliptical, and very regularly so; and hence we might expect the opposite hemisphere to be equally so, and to be a portion of the same curve. Nevertheless the degree measured at the Cape of Good Hope by **Lacaille**, in latitude 33° 18′ appears to indicate an ellipse of less eccentricity, or of greater axis; for the linear extent of 57037 toises, corresponds to the measure of a degree in latitude 47° 47′ in the northern hemisphere. If now we calculate the arc as before, with an oblateness of

$\frac{1}{320}$, and with the sides of Lacaille's triangles reduced to the meridian, we find it greater by 10″ than it was found to be by observations of the stars. An error of 10 seconds, by an astronomer so skilful and scrupulous as Lacaille, is too extraordinary to be admitted as probable. It is true, that there was a greater error well ascertained to have occurred in the measurement in Lapland, amounting to 13 seconds; but the academicians engaged in this undertaking were by no means equally conversant with observations as Lacaille.

There remains therefore but one method of removing all doubt on this subject, and this is to repeat and verify the measurement at the Cape, and, if possible, to extend it still farther to the north. The same Major Lambton, who has succeeded so well in Asia, and is in possession of such perfect instruments for the purpose, would be singularly qualified for a similar undertaking in Africa, and would furnish us with a measurement in the other hemisphere, as much to be relied upon as the former. He would have the glory of deciding two important questions by his own observations; first, the similarity and magnitude of the two hemispheres; and, secondly, the degree of reliance to be placed on the elliptic hypothesis.

It might be still further desirable, if other measurements could also be undertaken, either in New Holland, or in Brazil; for though neither of these countries differs much in latitude from the Cape of Good Hope, they are so remote in longitude, that a correspondence of measures so taken would nearly establish the similarity of all meridians.

Note.

I shall now explain the formulæ employed in deducing the results to which I have come in the foregoing Memoir. The demonstration of them is to be found in the work of M. DELAMBRE, on the Meridian.

In the first place, let a be the radius of the equator, e the eccentricity, ψ the latitude of one extremity of a side, or arc, in any series of triangles, and θ the azimuth of that side. The radius of curvature of this arc will be expressed by

$$\frac{1}{R1} = \frac{\left(1 + \frac{e^2}{1-e^2} . \cos.^2\psi . \cos.^2\theta\right)}{R} \text{ and } \frac{1}{R} = \frac{(1-e^2 . \sin.^2\psi)^{\frac{1}{2}}}{a}.$$

Hence we see that R is the radius of the arc at right angles to the meridian. One may in general neglect the azimuth, and take the last radius for the radius R1. Now, in computing the arc between Clifton and Dunnose, I have supposed the oblateness to be $\frac{1}{330}$ or $e^2 = \frac{669}{330^2}$, and log. $a = 6{,}5147200$ expressed in toises.

The latitude of the southern extremity of the base is the same as that of Clifton, and its azimuth, if we choose to attend to it, is nearly 335° 28′. This base, considered as an arc of a circle, is reduced to its sine by the formula $\varepsilon = \log. \varepsilon - \frac{K . \varepsilon^2}{6R^2}$, (K being the modules of the table of logarithms, so that log. K = 9,6377843.)

By means of the logarithmic sine of the base, and the angles of the triangles, considered as spherical, the logarithmic sines of the sides in the series were next computed, and then reduced to logarithms of the arcs themselves by the formula $\log.\ \epsilon = \log.\ \sin.\ \epsilon + \frac{K\ .\ \sin.^2 \epsilon}{6R^2}$.

For the purpose of making this last reduction, it is sufficient to take a single value of R, corresponding to the mean latitude of the entire arc 52° 2′ 20″. It was thus that the table was formed of logarithmic sides considered as arcs.

Let *m* be one of these arcs, and let us represent by $\delta\psi$ and $\delta\psi''$ its value reduced to the meridian, the one in toises, the other in seconds of a degree, and we shall have the following formulæ;

$$\delta\psi = m\ .\ \cos.\ \theta - \left(\frac{m^2\ .\ \sin.^2\theta}{2R}\right).\ \text{tang.}\ \psi - \left(\frac{m^2\ .\ \sin.^2\theta}{2R}\right).\left(\frac{m\ .\ \cos.\ \theta}{3R}\right).\ (1 + 3\ .\ \tan.^2\psi)$$

$$\delta\psi'' = \left(\frac{\delta\psi}{R\ .\ \sin.\ 1''}\right) + \left(\frac{\delta\psi}{R\ .\ \sin.\ 1'}\right).\ e^2.\ (1 + e^2)\ .\ \cos.^2\psi\ .\ \left\{1 \mp \left(\frac{3\ \tan.\ \psi}{2}\right).\left(\frac{\delta\psi}{R}\right)\right\}$$

: the superior sign being taken when the latitude ψ'' is greater than ψ, and the inferior when it is less.

The correction dependent on the convergence of the meridian for the azimuths is $\delta\theta = \left(\frac{m\ .\ \sin.\ \theta}{R1\ .\ \sin.\ 1''}\right).\left(\frac{\sin.\ \frac{1}{2}(\psi + \psi')}{\cos.\ \psi'\ .\ \cos.\ \frac{1}{2}\ \delta\psi''}\right)$.

Hence the azimuth of the first station seen from the second and reckoned westward from the north, is $\theta' = 180° + \theta + \delta\theta$.

If P'' be put for the difference of longitude between two points distant by an arc which measures *m*, we have $\sin.\ P'' = \frac{\sin.\ m\ .\ \sin.\ \theta}{\cos.\ \psi}$, $\log.\ \sin.\ m = \log.\ \left(\frac{m}{R1}\right) - \frac{K}{6}\ .\ \left(\frac{m}{R1}\right)^2$, and $\log.\ P'' = \log.\ \left(\frac{\sin.\ P''}{\sin.\ 1''}\right) + \frac{6}{K}\ .\ (\sin.\ P'')$.

The arc of the meridian, between Greenwich and Formentera, is so fortunately situated, that its middle point is in latitude 45°. Its whole extent measures 12° 48′ 44″, and the distance between the parallels, in linear measure, was found to be 730430,7 toises. Hence the mean degree, corresponding to the latitude of 45° 4′ 18″, is 57010,5 toises; and if we multiply this number by 90°, we get one-fourth part of the meridian of the earth.

The correction to be deduced for oblateness is 58, 59, or 61 toises, according as it is assumed to be $\frac{1}{330}$, $\frac{1}{320}$, or $\frac{1}{310}$, and if we take the mean of these, we have the fourth part of the meridian $Q = 5130886$ toises; and hence the metre = 44330867 lines; so that the value of the metre turns out to be almost entirely independent of the elliptical form of the earth.

The radius of the equator is derived from the expression $\log. a = \log. \left(\frac{2Q}{\pi}\right) + K . (\frac{1}{2} . \epsilon + \frac{1}{16} . \epsilon^2 - \frac{1}{48} . \epsilon^3)$, ϵ being the oblateness, and π the periphery of a circle $= 3,1416$.

In order to compare any degrees measured with those obtained on the elliptic hypothesis, we have a very simple formula. Let m and m' be the values of two degrees on the meridian, of which the mean latitudes are $\psi 1$ and $\psi 2$; in comparing the analytic expressions for these two degrees, developing them, and then making $\psi = 45°$, we have $m' = m . (1 - \frac{1}{2} . p . \cos. 2\psi 2 + g . \cos.^2 2\psi 2)$, $m = 57010,5$ toises, $p = \frac{3}{2} e^2 . (1 + \frac{1}{2} e^2) . \frac{\sin. 1°}{1° . \sin. 1''}$, and $g = \frac{15}{64} . e^4 . \left(\frac{\sin. 2°}{1° . \sin. 1''}\right)$.
And then we shall find that the oblateness $\frac{1}{320}$ gives 57075,66 and 57192,38 toises for the degrees in England and Lapland.

I shall here subjoin one reflection more, which appears of importance. The oblateness of the earth is a quantity which varies considerably, by the least difference in the elements on which it depends. Accordingly it is not surprising, that its value fluctuates between two proportions which differ sensibly from each other. To illustrate this, let p be the function which serves to determine the oblateness of the earth, so that $\frac{1}{\varepsilon} = p$. When this equation varies $-\delta\varepsilon = \varepsilon^2 . \delta p$.

Now the coefficient ε^2 being very great, we see why the least variation in the elements of the function p, occasions so considerable a variation in the denominator of the oblateness. This is precisely what happens in the lunar equations dependent on the figure of the earth, and which M. Laplace has deduced from his beautiful theory. Thus, for example, in the inequality that depends on the longitude of the moon's node, which he has determined analytically with so much precision, the numerical coefficient found by Burg gives $\frac{1}{305}$ for the oblateness; but if this coefficient be diminished by 0″,665, then the oblateness becomes $\frac{1}{320}$, so that a variation even to this small amount in the coefficient augments the denominator of the oblateness nearly $\frac{1}{20}$ part.

The same happens with regard to the pendulum vibrating seconds; for, supposing its length at 45° to have been correctly ascertained by M M. Biot and Mathieu, if we wish to know the length of a second's pendulum at the equator, corresponding to an oblateness of $\frac{1}{310}$, we find it to be 439,1810 lines. Now this length differs from that determined by Bouguer only by 0,029 of a line, and M. Laplace even thinks that the result of Bouguer should be diminished by about double this quantity. We see from hence how much these little differences, whether produced by errors of observation, or irregularities in the earth itself, are liable to affect the denominator of the fraction expressing the oblateness.

Fortunately, it seems probable, that the utmost latitude of our present uncertainty is between the limits of 330 and 310, and the mean of these may be considered as a very near approximation to the truth.

XVIII. *An Account of some Experiments on different Combinations of Fluoric Acid. By* John Davy, *Esq. Communicated by Sir* Humphry Davy, *Knt. LL.D. Sec. R. S.*

Read June 11, 1812.

Introduction.

Two years ago, I engaged, at the request of my brother, Sir H. Davy, in an inquiry respecting the nature of common fluoric acid gas. My principal object was to ascertain whether silex is essential to its constitution, and whether the proportion is constantly the same. This subject, and experiments on the fluoric and fluoboracic acids, occupied me for about six months. Since that time, the work of M. M. Gay Lussac and Thenard has appeared, entitled "Recherches Physico-Chemiques," in the second volume of which is an elaborate dissertation on fluoric acid. These philosophers, I find, have anticipated many of my results, and consequently very much abridged my labour of detail in the following pages. To repeat what is already known would be useless, I shall therefore confine myself to describe what I have observed, which appears to me yet novel, or different from the observations of the French chemists. The order which I shall pursue, will be that which I observed in my experiments. I shall divide what I have to advance into four parts. The first part will relate to the silicated fluoric acid gas, and to the subsilicated fluoric acid; the second to the combinations of these acids, and of pure fluoric acid with ammonia; the third to fluoboracic acid; and the fourth to its ammoniacal salts.

SECT. 1. *On silicated fluoric acid Gas, and subsilicated fluoric Acid.*

The facts which have already been published by M. M. GAY LUSSAC and THENARD and others, appear to me to be sufficient to prove that pure fluoric acid has not yet been obtained in the gaseous state, and that silex, or boracic acid, is requisite that it may assume this form. Were more evidences necessary, I could advance many in point. One circumstance only I shall mention, proving that common fluoric acid gas is perfectly saturated with silex. I have preserved this gas, made by heating, in a glass retort, a mixture of fluor spar and sulphuric acid, for several weeks over mercury in a glass receiver uncoated with wax, without observing the slightest erosion to be produced.*

This gas, with great propriety, has lately been called silicated fluoric. Before I proceed to its analysis, I shall notice what method I have found the best for obtaining it. I have, for a considerable time, long before M. M. GAY LUSSAC and THENARD's work was published, added to the mixture of fluor spar and sulphuric acid, a quantity of finely pounded glass, and have thus procured the gas with the greatest facility. The advantages of this addition are considerable. The retort is saved, which otherwise, in less than one operation, would be destroyed; and a much larger quantity of gas is procured from the same materials, and with less trouble and less heat; the action indeed, at first is so powerful, that gas begins to come over before the application of heat is made, and a very gentle one only is required to continue its production.

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* The sides of the receiver indeed became obscure; but this was not from erosion, but from deposition, as appeared from the transparency and polish of the glass being readily restored by slight friction. What the deposition was, I am ignorant of. After several weeks it was so trifling, as to give only a slight degree of opacity to the receiver.

Previous to its analysis, it was necessary to ascertain the specific gravity of the gas. This I have endeavoured to do. The gas, the subject of experiment, was quite pure, being totally condensed by water. A Florence flask was exhausted; in this state, weighed by a very delicate balance, it was

= 1452.2 grains.
Filled with common air - = 1452.2 + 10.2
Again exhausted - - = 1452.2
Filled with silicated fluoric gas = 1452.2 + 36.45

Hence as 10.2 : 31 : : 36.45 : : 110.78.

Thus it appears, that 100 cubic inches of silicated fluoric acid gas, at ordinary temperature and pressure, are equal to 110.78 grains.

When silicated fluoric acid gas is condensed by water, it is well known that part only of the silex is deposited. To obtain the whole, in order to ascertain the proportion in the gas, I have employed ammonia in excess. 40 cubic inches of the gas (barom. 30, therm. 60) were transferred in portions of 10 cubic inches, at a time to a solution of ammonia. The silex precipitated was carefully collected on a filter, and washed till the water that passed through it, ceased to be affected by nitrat of lime. It was next dried, and strongly heated in a platina crucible. It weighed 27.2 grains, and was pure silex. Supposing fluoric acid to be the remaining 17.1 grains, which added to 27.2 grains are equivalent to the weight of 40 cubic inches of the gas, it appears that 100 parts by weight of this gas consist of

61.4 silex
38.6 fluoric acid
100,0

That this estimate may be correct, it is evident, that ammonia should have the property of precipitating the whole of the silex of silicated fluoric gas; which I shall not now endeavour to prove, but leave it to be considered in another part of the paper.

There is no improbability attached to the idea, that silicated fluoric acid gas may, from the manner in which it is prepared, contain a proportion of alkali. To discover whether this was the case, a solution of nitrat of lime was added to the ammoniacal solution neutralized by nitric acid, from which the silex in the preceding experiment had been removed. The precipitate of fluat of lime was separated by filtration. The filtered liquid was evaporated to dryness; and the ammoniacal salt heated in a platina crucible till it was entirely dissipated. The residue had the appearance and taste of quick lime. It was dissolved in acetic acid, and the solution yielded sulphat of lime on the addition of sulphat of ammonia. The liquid was evaporated to dryness, and when the residuum had been heated to dull redness, nothing remained but a little white powder, weighing about a grain, and having all the properties of gypsum. Thus it appears that silicated fluoric acid gas contains no alkali.

My next object was to ascertain the composition of common liquid fluoric acid—that acid obtained by the decomposition of silicated fluoric acid gas by water, and which, on account of the separation that occurs of part of the silex, may, with greater propriety, be called subsilicated fluoric acid. For this purpose, 43.21 cubic inches, barom. 30.4, therm. 50, or 44 cubic inches at common temperature and pressure, were successively added, two cubic inches at a time, to one cubic inch of distilled water in a small jar over mercury. The whole of this, the gas being pure, was readily condensed. The temperature was somewhat raised. The silex precipitated, formed a gelatinous mass of a blueish colour, which had absorbed all the water like a sponge, so that none appeared fluid. This gelatinous mass was carefully transferred to a filter, and washed with distilled water till it was rendered insipid and incapable of reddening litmus paper. It retained its blueish hue only whilst moist. When dried and ignited, it was in thin lamellæ, and of a snow-white colour, and surprisingly bulky. It weighed 7.33 grains, and was found to be pure silex. Thus it appears that the subsilicated fluoric acid formed by the decomposition of 44 cubic inches of silicated fluoric acid gas contains 7.33 grains of silex less than the gas itself. Consequently independent of water, which no doubt is essential to this acid, 100 parts of it seem to consist of

54.56 silex
45.44 acid
100.00

I have endeavoured to ascertain what quantity of silicated fluoric acid gas a given quantity of water will condense. In one instance $\frac{19}{100}$ of a cubic inch of distilled water absorbed 51 cubic inches, barom. 30.5, therm. 60. The gas was added to the water in a jar over mercury, as fast as it was absorbed. The experiment was stopped, when the gas, after having remained in contact with the water a whole night, ceased to be diminished. According to this result, the proper correction being made for the additional pressure, water decomposes about 263 times its bulk of silicated fluoric acid gas.

Dr. Priestley observed, that muriatic acid gas reproduced silicated fluoric gas from the crust of silex formed, when the latter is condensed by water.* This experiment I have repeated, and as it appears to show more correctly the quantity of gas water can condense, I shall describe the result. 2.4 cubic inches of muriatic gas were added to a drop of water, that had previously absorbed one cubic inch of silicated fluoric gas, in a jar over mercury. There was an immediate absorption equal to $\frac{2}{10}$ of a cubic inch. The mixture of silex and subsilicated fluoric acid effervesced, and from an apparent solid became fluid, the whole of the silex gradually disappearing. After the first mentioned absorption, there was no farther. The gas produced was silicated, as appeared from the crust it deposited when removed to water, and the liquid formed was pure muriatic acid, for decomposed by concentrated sulphuric, it afforded merely muriatic acid gas, without any silicated fluoric. The evident conclusion from the preceding result is, that water condenses equal quantities of the muriatic and silicated fluoric acid gasses, and consequently that the first estimate is too low, and instead of 263 times its bulk, it is probably more correct to say that water to be saturated requires at least 365 times its volume. Neither will this estimate appear inconsistent with the former result, when the deposition of silex is considered as an obstacle to the free exposure of the surface of the water to the gas.

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* Vide Priestley on Air, Vol. II. p. 202.

Subsilicated fluoric acid is decomposed by ammonia and the fixed alkalies, and by all the earths that I have made trial of. It is also decomposed by the sulphuric acid and the boracic, as well as by the muriatic acid gas.

Of the particular changes which occur when it is acted upon by the alkalies, I defer giving any account at present, as it is my intention to do it in the next section.

To learn the effect of heat on it, a small quantity of strong acid, pure and transparent was introduced into a retort connected with mercury. A spirit lamp being applied about three cubic inches of silicated fluoric acid gas were produced. The neck of the retort was lined with silex in a gelatinous state, and much liquid subsilicated fluoric acid, that had distilled over, was condensed in the colder part of the neck, and was absorbed by bibulous paper previously introduced, to prevent the distilled fluid from entering the jar for the reception of the gas. When the whole of the acid in the bulb of the retort had been evaporated, little or no silex remained.

The general result of this experiment is very different from that which Dr. PRIESTLEY, who first made it, obtained. Instead of silicated fluoric acid gas, he procured "vitriolic acid air," sulphureous acid gas.

I have tried also the effect of heat on the silicious crust, formed by the decomposition of silicated fluoric acid gas, by water; but could obtain no sulphureous acid gas, as Dr. PRIESTLEY did only a small quantity of silicated fluoric.

The correctness of Dr. PRIESTLEY's observations cannot be doubted. I can only account for his results, by supposing that some sulphuric acid in consequence of the high temperature employed in making the gas was volatilized, and mixed with the subsilicated fluoric acid, and that mercury also was present from the acid being prepared over this metal.

These experiments too oppose another statement relative to a method prescribed for making fluoric acid gas free from silex, by merely heating strong subsilicated fluoric acid in a retort, and collecting the gas over mercury. It is asserted, in chemical works of some reputation, that this process is successful. I have never found it so, having always obtained results similar to those above stated. This, I suppose, is one of the many errors that have secretly crept into repute, and has been believed, because never subjected to the test of experiment.

The action of concentrated sulphuric acid on subsilicated fluoric acid, is similar to that of muriatic acid gas, occasioning a disengagement of silicated fluoric acid gas. Facts which appear to prove, that water is absolutely essential to the existence of this acid.

Boracic acid decomposes it, in a very different way, not from any predominant affinity for the water, but in consequence of a stronger attraction for the fluoric acid itself. Silicated fluoric acid of course is not produced; but liquid fluoboracic acid and the silex is precipitated in a gelatinous state, as when ammonia is employed.

These are the principal facts I have to notice respecting this acid. Before I conclude, I shall briefly mention a few other circumstances. Applied to the tongue, in its concentrated state, it produces a very painful sensation, like that which strong muriatic acid does, and it has a very similar effect on the cuticle. It does not appear to erode glass, for I have kept it in bottles of this substance more than a month without any action being perceptible. Exposed to the air, it slowly and almost completely evaporates, there being only a very trifling silicious residue; and when gently heated in an open vessel, it is rapidly dissipated in white fumes.

SECT. II. *On the Combinations of silicated fluoric acid Gas, and the subsilicated Fluoric, and the fluoric Acids with Ammonia.*

M. GAY LUSSAC has shewn that silicated fluoric acid gas, like carbonic acid gas, condenses twice its volume of the volatile alkali.* The experiment I have several times repeated, and constantly with the same result, no difference appearing when the acid gas was added in great excess to the alkaline, or the alkaline to the acid. This being the case, and knowing the specific gravities of the two gasses,† 100 parts by weight of silicated fluat of ammonia seem to consist of

24.5 ammonia
75.5 acid

100.0

Silicated fluat of ammonia volatilizes unaltered, if heated by a spirit-lamp in the vessel in which it is formed, and provided moisture be entirely excluded.

Like silicated fluoric acid gas itself, this salt is decomposed by water, and a similar precipitation of silex occurs, and in the same proportion. Thus the salt formed by the union of 30

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[*] Vide Mém. d'Arcueil, Tom. II.

[†] According to Sir H. DAVY, 100 cubic inches of ammonia, barom. 30, therm. 60, weigh 18 grains. It is this estimate which I have taken.

cubic inches of silicated fluoric gas, and 60 of volatile alkali (barom. 30, therm. 60) in a small glass jar over mercury, being carefully collected and introduced into water, afforded five grains of pure silex, weighed after being well washed and heated to redness.

The saline solution, since part of the silex of the silicated fluoric acid gas is separated during its production, appears to be a subsilicated fluat, or a combination of subsilicated fluoric acid and ammonia. Another mode of making it, more directly proves that this is its composition. When ammonia is added to the subsilicated fluoric acid in excess, this salt is formed without any precipitation. From these facts, it may be concluded, that independent of water, which appears to be essential to its existence, 100 parts of it consist of

28.34 ammonia
71.66 acid

100.00

Subsilicated fluat of ammonia has a pungent saline taste. It just perceptibly reddens litmus paper. Slowly evaporated, it forms small transparent and brilliant crystals. The largest I could obtain, appeared to be tetrahedral prisms. The solid salt is very soluble in water; but is not deliquescent. When heated it appears to sublime unaltered. It is curious that the solution of this salt, when evaporated by a heat near its boiling point, powerfully erodes the glass or porcelain vessel, and a residuum of silex appears, on the addition of water, to redissolve the salt. This erosion and residue of silex I have seen produced three times following, with the same quantity of salt. I mention the fact, which, I believe, was before observed by Scheele, without attempting an explanation of it. It may perhaps be said, that as the water evaporates, the affinity of the subsilicated fluat for silex increases.

Subsilicated fluat of ammonia is decomposed by the sulphuric acid, and by muriatic acid gas, and also by the fixed alkalies and by ammonia.

Sulphuric acid expels from it, silicated fluoric gas and hydrated fluoric acid fumes.

Muriatic acid gas acts slowly on it, and effects its decomposition apparently through the medium of its water. A little of the crystalline salt was introduced into muriatic acid gas in a jar over mercury. In a short time some silicated gas was produced, as the silicious deposition, on the addition of water, indicated. Strong muriatic acid was substituted for the acid gas. Now no apparent change took place, for on evaporating the acid, the residue, decomposed by sulphuric acid, afforded only silicated fluoric acid gas.

The alkalies form by the decomposition of this salt, the same compounds that they do by their action on subsilicated fluoric acid.

Potash expels the ammonia, and produces the silicated fluat and fluat of potash, as M. M. GAY LUSSAC and THENARD have described.

The changes occasioned by soda appeared to me similar; but the gentlemen just mentioned, assert that this alkali precipitates the whole of the silex, and does not form a triple salt with it and part of the acid.

Ammonia seems to me to separate completely the silex, and by uniting with the pure acid to constitute a true fluat. MM. GAY LUSSAC and THENARD are of a different opinion. They say that the whole of the silex cannot by this method be removed, but only the principal part. Their reason for this belief, is, that on repeatedly evaporating the salt after the addition of ammonia and redissolving it, they have each time observed a residue of silex. If they employed metallic evaporating vessels, the results of my experiments do not agree with theirs; for making use of platina for this purpose, and adding an excess of ammonia, I never detected traces of silex on evaporating the filtered fluat. But our results agree, if they employed glass or porcelain vessels, which fluat of ammonia has the property of corroding.

I now proceed to the consideration of fluat of ammonia; but before I describe some of the properties of this fluat which I have observed, I shall briefly mention the means pursued for ascertaining the proportions of its constituent parts.

The composition of subsilicated fluat of ammonia being known, that of the fluat (granting what is already advanced respecting its formation to be correct) may be inferred from the proportion of silex, that a given quantity of ammonia will precipitate. 18 cubic inches of ammoniacal gas were condensed by $\frac{1}{4}$ of a cubic inch of distilled water in a small glass tube over clean mercury. This ammoniacal solution was added to a clear filtered solution of subsilicated fluat of ammonia. A precipitate of silex was immediately produced. After several hours standing, this precipitate was collected on a filter, well washed and dried and heated to redness. It was pure silex, and weighed 1.6 grains. This experiment, like all the preceding, was repeated, and the result confirmed. In both instances there was an excess of subsilicated fluat. The precipitations were made in a platina vessel, and the solutions were neither heated before or after the separation of the silex. Calculating from this result, 100 parts of fluat of ammonia seem to consist of

76.4	ammonia
23.6	fluoric acid
100.0	

Water appears to be a constituent part of this salt.

It may be rendered neutral by means of a gentle heat, which expels the excess of ammonia employed in its formation. In its neutral state, it has a strong saline taste, and it readily deliquesces when exposed to the atmosphere. Like the neutral carbonats, it is decomposed by heat; but there is this difference between them, part of the pure alkali is expelled instead of the acid, and an acid fluat of ammonia is formed. A gentle heat only is required for the purpose, that of boiling water is nearly sufficient. When the heat is much stronger, the salt fuses and passes off in dense fumes of a most peculiar suffocating odour. The effects of these fumes, when inhaled, are very powerful and disagreeable, and even dangerous, I might venture to say, were I to speak from my own experience. In one instance, when I inhaled only a small quantity, they produced in a few minutes a violent cough and catarrh, and apparent accumulation of blood in the neck and head, and symptoms altogether not unlike those the attendants of apoplexy, which continued for about a quarter of an hour, and then slowly diminished, and gradually disappeared without leaving any permanent bad effect. The fluat of ammonia, when heated in a metallic vessel, appears to sublime unaltered. But the result is different when the experiment is made in a glass one. Ammonia is expelled, the glass is corroded, and subsilicated fluat of ammonia is formed and sublimed. Its action on glass is so powerful, that I have successfully employed it instead of fluoric acid itself, for etching on this substance. It has one advantage, that it is more manageable. The solution may be applied by means of a hair pencil or a common pen to the glass, and the erosion will be produced by exposure to a moderate temperature.

The fixed alkalies, and all the earths that I have tried, decompose this salt; they expel the ammonia, and form true fluats with the acid itself. I have examined all the fluats thus formed, and have endeavoured to ascertain the proportions of their constituent parts; but I am not sufficiently satisfied of the accuracy of the results, to venture to give an account of them.

SECT. III. *On Fluoboracic Acid Gas.*

MM. GAY LUSSAC and THENARD, who first discovered this gas, obtained it by heating strongly, in an iron tube, a mixture of fluor spar and fused boracic acid. I have found that it may be more easily procured, in greater abundance, and at less expence, by gently heating, in a common glass retort, a mixture of finely pounded boracic acid * and fluor spar with concentrated sulphuric acid. 1 part by weight of fused boracic acid, 2 parts of fluor spar, and about 12 of sulphuric acid appear to be the proportions best adapted for the purpose. This method will require no explanation when it is considered that boracic acid, as has already been observed, precipitates silex from liquid subsilicated fluoric acid. If the heat is gentle, not nearly sufficient to occasion the ebullition of the sulphuric acid, and the proportions just recommended are used, the retort will not be injured, and pure fluoboracic acid gas will be produced in abundance. When the gas ceases to come over, if the heat is raised, more will be evolved, and there will be distilled over at the same time, a viscid fluid, which is a compound of sulphuric acid and fluoboracic acid gas. Now the operation should be stopped, if the object is to obtain merely pure fluoboracic gas, a long continuation of the heat producing some silicated fluoric. Before quitting the subject, it should be observed that the quantity of sulphuric acid employed is of considerable consequence to the success of the experiment. If too much is used, there is a great loss of gas from the property which sulphuric acid has of absorbing fluoboracic acid gas; and if too little is employed, it soon becomes diluted, and loses the power of generating the gas, though it may still decompose the fluor spar. Both extremes, therefore, are to be avoided, and the proportion of acid mentioned above, as far as my experience goes, appears to be the best.

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* Common calcined borax answers the same end, but no so well. Its only recommendation to preference is cheapness.

I have endeavoured to ascertain the specific gravity of fluoboracic gas.

The flask exhausted weighed 1400.5 grains.
Filled with common air - 1400.5 + 6.2
Again exhausted - - 1400.5
Filled with pure acid gas - 1400.5 + 14.7.

Thus it appears that 100 cubic inches of fluoboracic gas are equal to 73.5 grains.

MM. Gay Lussac and Thenard have described the compound of this gas and water, a fuming fluid, in many respects similar to concentrated sulphuric acid. Like this acid, I have observed that it possesses a slight degree of tenacity, so that it has an oily appearance when poured from one vessel to another; and similar in another respect, it possesses the property of charring animal and vegetable substances, and which the French chemists observed belonging to the gas itself. I have found that water condenses more of this, than it does of any other known gas, no less than 700 times its volume. The experiment was then made, barom. 30.5, therm. 50, $\frac{14}{100}$ of a cubic inch of water were introduced into a tube over mercury, and the gas, in portions of 5 cubic inches at a time, was added until 100 cubic inches had been absorbed, when the water was apparently saturated. This acid was of the specific gravity 1.77.

The property which sulphuric acid has of absorbing fluoboracic acid gas has already been noticed. I found that ½ cubic inch of sulphuric acid, of the specific gravity 1.85, condensed 25 cubic inches of the gas, or 50 times its volume. The compound acid was strongly fuming, and appeared more tenacious than pure sulphuric acid, yet not nearly so much so as that compound of the two which distills over during the latter part of the operation of making fluoboracic gas.

This latter compound has some peculiarities. It is so tenacious, that it flows very slowly. It appears to be far more volatile than pure sulphuric acid. When poured into water, a dense white precipitate is formed, the exact nature of which I have not yet satisfactorily ascertained; but which is not produced by the direct compound of sulphuric acid, and the fluoboracic.

SECT. IV. *On the Combinations of fluoboracic acid Gas, and ammoniacal Gas.*

M. GAY LUSSAC has combined fluoboracic acid gas with ammonia. He states, that equal volumes of the two gasses condense each other.[*] This I have found to be the case, and I have also found that fluoboracic acid gas condenses twice, and even three times its volume of the volatile alkali. The compound observed by M. GAY LUSSAC is solid, white, and opaque, like the ammoniacal salts. The combinations I have obtained are liquid, transparent, and colourless, like water, though they are entirely free from this fluid. They were made by the direct union of the two gasses. 5 cubic inches of ammoniacal gas were added to the same volume of the fluoboracic gas contained in a small jar over dry mercury. There was a complete condensation of both, and the solid salt was the result. 5 cubic inches more of ammonia were introduced. The whole was quickly absorbed, and the solid salt was converted into the transparent fluid. 5 cubic inches more were added, which too were slowly absorbed, but without any change of form.

The solid salt volatilizes in close vessels unaltered, on the application of a gentle heat.

Both fluid compounds, when heated, are rendered solid, from the expulsion of part of the ammonia. Exposure to the air is attended with the same change, and the same effect is produced by the muriatic and carbonic acid gasses.

Knowing the volumes of the acid, and alkaline gasses which combine, it is easy to calculate the proportions of each by weight in the respective salts.

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[*] Vide Mém. d'Arcueil, Tom. II.

100 Parts consist of	Ammonia.	Acid.
The solid compound	19.64	80.32
The first fluid -	32.9	67.1
The second fluid	42.4	57.6

These combinations are curious in many points of view. They are the first salts that have been observed liquid, at the common temperature of the atmosphere, without containing water. And they are additional facts in support of the doctrine of definite proportions, and of the relation of volumes.

XIX. *On a Periscopic Camera Obscura and Microscope.* By William Hyde Wollaston, *M. D. Sec. R. S.*

Read June 11, 1812.

Although the views, which I originally had of the advantage to be derived from the periscopic construction of spectacles,[*] naturally suggested to me a corresponding improvement in the *camera obscura*, by substituting a meniscus for the double convex lens, I have hitherto deferred making it known to others, except as a subject of occasional conversation.

Since in vision with spectacles, as in common vision, the pencil of rays received by the eye in each direction is small, the superiority of that form of glass, which disposes all parts of it most nearly at right angles with the visual ray, admits of distinct demonstration; but with respect to the camera obscura, where the portion of lens requisite for sufficient illumination, is of considerable magnitude, although it is evident that some improvement may be made in the distinctness of oblique images on the same principles, yet as the focus of oblique rays is far from being a definite point, the degree in which it may be improved is not a fit subject of mathematical investigation.

I have therefore had recourse to experiments, in order to determine by what construction the field of distinct representation may be most extended; and, I trust, the result will be acceptable to this Society. I shall take the same opportunity to describe an improvement in the construction of the simple microscope, which may also be termed periscopic, as the object of it is to gain an extension of the field of view, upon the same principles as in the preceding instances, namely, by occasioning all pencils to pass as nearly as may be at right angles to the surfaces of the lens. The mode, however, in which this is effected is apparently somewhat different in the practical execution.

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[*] Phil. Magaz. Vol. XVII. Nicholson's Journal, VII. 143.

In the common *camera obscura*, where the images of distant objects are formed on a plane surface to which the lens is parallel, if the surfaces of the lens be both convex, and equally curved (as in fig. 1); and if the distance of the lens be such, that the images formed in the direction of its axis CF be most distinct, then the images of lateral objects are indistinct in a greater or less degree, accordingly as they are more or less remote from the axis. The causes of this indistinctness may be considered as twofold; for in the first place, all parts of the plane, excepting the central point, are at a greater distance from the centre of the lens than its principal focus; and secondly, the point *f*, to which any pencil of parallel rays passing obliquely through the lens are made to converge, is less distant than the principal focus. On this account, it is in general best to place the lens at a distance somewhat less than that which would give most distinctness to the central images, because in that case a certain moderate extension is given to the field of view, from an adjustment better adapted to lateral objects, without materially impairing the brightness of those in the centre. The want of distinctness, however, is even then only diminished in degree, but is not remedied.

The construction, by which I propose to obviate this defect, is represented in the second figure, in which are seen the essential parts of a periscopic camera in their due proportion to each other. The lens is a meniscus, with the curvatures of its surfaces about in the proportion of two to one, so placed that its concavity is presented to the objects, and its convexity toward the plane on which the images are formed. The aperture of the lens is four inches, its focus about twenty-two. There is also a circular opening, two inches in diameter, placed at about one-eighth of the focal length of the lens from its concave side, as the means of determining the quantity and direction of rays that are to be transmitted.

The advantage of this construction over the common camera obscura is such, that no one who makes the comparison, can doubt of its superiority; but the causes of this may require some explanation. It has been already observed, that by the common lens, any oblique pencil of rays is brought to a focus at a distance less than that of the principal focus. But in the construction above described, the focal distance of oblique pencils is not merely as great, but is greater than that of a direct pencil. For since the effect of the first surface is to occasion divergence of parallel rays, and thereby to elongate the focus ultimately produced by the second surface, and since the degree of that divergence is increased by obliquity of incidence, the focal length resulting from the combined action of both surfaces will be greater than in the centre, if the incidence on the second surface be not so oblique as to increase the convergence. On this account, the opening E is placed so much nearer to the lens than the centre of its second surface, that oblique rays E*f*, after being refracted at the first surface, are transmitted through the lens nearly in the direction of its shorter radius; and hence are made to converge to a point so distant that the image (at f) falls very nearly in the same plane with that of an object centrally placed.

In the use of spectacles by long-sighted persons, the course of the rays in the opposite direction is so precisely similar, that the same figure might serve to illustrate the advantages of the periscopic construction. For the purpose of seeing the extended page of a book (as at AB) with least fatigue to the eye, that form of lens will be most beneficial, which renders the rays received from each part of its surface parallel; and this is effected by the exact counterpart to the preceding arrangement; for in this case the opening E represents the place of the eye receiving parallel rays from the lens in each direction, instead of transmitting them from a distance towards it.

There is, however, this difference between the two cases, that in the camera obscura a much larger portion of the lens is required to conspire in giving a distinct image of any one object; so that the conformation best adapted for lateral objects, would not be consistent with distinctness at the centre; and hence arises a limit to the application of the principle. On the common construction, the whole lens is so formed as to give brilliancy and distinctness at the centre alone, without regard to lateral objects. In adopting such a deviation from the customary form, as I propose, in favour of a more extended view, some diminution of the aperture is required in order to preserve the desired distinctness at the centre. In my endeavours to ascertain the most eligible form of meniscus for this purpose, I have assumed sixty degrees to be the field of view required. But when so large a field is not wanted, then a lens that is less curved will be preferable; and the proportion of the radii must be varied according to the angular extent intended to be included.

For the purpose of estimating by what combination of radii any required focal length may be given to a meniscus, I have contrived a diagram by which very much labour of computation may be saved, as a very near result may be obtained by mere inspection. This contrivance is founded on the well known formula for the focal length of any lens $F = \frac{mrR}{R \pm r}$: m being a certain multiple obtained by dividing the sine of refraction by the difference of the sines of incidence and refraction. Hence, in applying this formula to the meniscus, $F : R :: mr : R - r$. In fig. 3, lines expressive of these quantities are so arranged, that by assuming any point F corresponding to the focal length desired, and drawing a line FR through a point R indicating any supposed length of the greater radius, the corresponding length of the other radius will be found where the line drawn intersects the middle line in the diagram.

In laying down these lines, the length and position of AF and AR were assumed at pleasure; and they were divided into any number of equal parts. But the position and length of the middle line Ax was adapted with care to the refractive power of plate glass in the following manner. Since $m = \frac{1}{1{,}505 - 1} = 1{,}98$, a line BC was drawn from the point 10 in the line AR, parallel to AF, and equal to 19,8 divisions of the primary lines; so that if r be $= 10$, then the line BC $= mr$. The distance AC being then divided into ten equal parts, with their subdivisions, afforded the means of continuing the same scale to any desired length. Since the first line BC was laid down parallel to AF, and equal to mr, any other lines drawn through corresponding numbers 7 and 7, 8 and 8, &c. will be also parallel, and by preserving due proportion, will correctly represent mr. Hence in all positions of the line FR, the same similarity of triangles obtains, and the same proportion of $F : R :: mr : R - r$; and consequently the focal length, corresponding to any assumed radii, is truly ascertained.

For the purpose of duly proportioning the curvatures of flint-glass, a second line Ay might be laid down in a mode similar to the preceding, by adapting the multiple $m = \frac{1}{1{,}58 - 1} = \frac{19}{11}$ to the different density of this glass.

With respect to the construction of a microscope on periscopic principles, I believe the contrivance to be equally new with the former, and equally advantageous. The great desideratum in employing high magnifiers is sufficiency of light; and it is accordingly expedient to make the aperture of the little lens, as large as is consistent with distinct vision. But if the object to be viewed, is of such magnitude as to appear under an angle of several degrees on each side of the centre, the requisite distinctness cannot be given to the whole surface by a common lens, in consequence of the confusion occasioned by oblique incidence of the lateral rays, excepting by means of a very small aperture, and proportionable diminution of light.

In order to remedy this inconvenience, I conceived that the perforated metal, which limits the aperture of the lens, might be placed with advantage in its centre; and accordingly I procured two plano-convex lenses ground to the same radius, and applying their plane surfaces on opposite sides of the same aperture in a thin piece of metal (as is represented by a section, fig. 4), I produced the desired effect; having virtually a double convex lens so contrived, that the passage of oblique pencils was at right angles with its surfaces, as well as the central pencil. With a lens so constructed, the perforation that appeared to give the most perfect distinctness was about one-fifth part of the focal length in diameter; and when such an aperture is well centered, the visible field is at least as much as twenty degrees in diameter. It is true, that a portion of light is lost by doubling the number of surfaces; but this is more than compensated by the greater aperture, which, under these circumstances, is compatible with distinct vision.

Beside the foregoing instances of the adaptation of periscopic principles, I should not omit to notice their application to the camera lucida; as there is one variety in its form, that was not noticed in the description which I originally gave of that instrument.*

In drawing, by means of the camera lucida, distant objects are seen by rays twice reflected (*d*, fig. 5), at the same time and in the same direction that rays (*e*) are received from the paper and pencil by the naked eye. The two reflections are effected in the interior of a four-sided glass prism, at two posterior surfaces inclined to each other at an angle of 135 degrees. In the construction formerly described, the two other surfaces of the prism are both plane, through which the rays are simply transmitted at their entrance and exit. But since an eye that is adjusted for seeing the paper and pencil, which are at a short distance, cannot see more distant objects dis-

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* Nicholson's Journal, XVII. p. 1. Phil. Magaz. XXVII. p. 343.

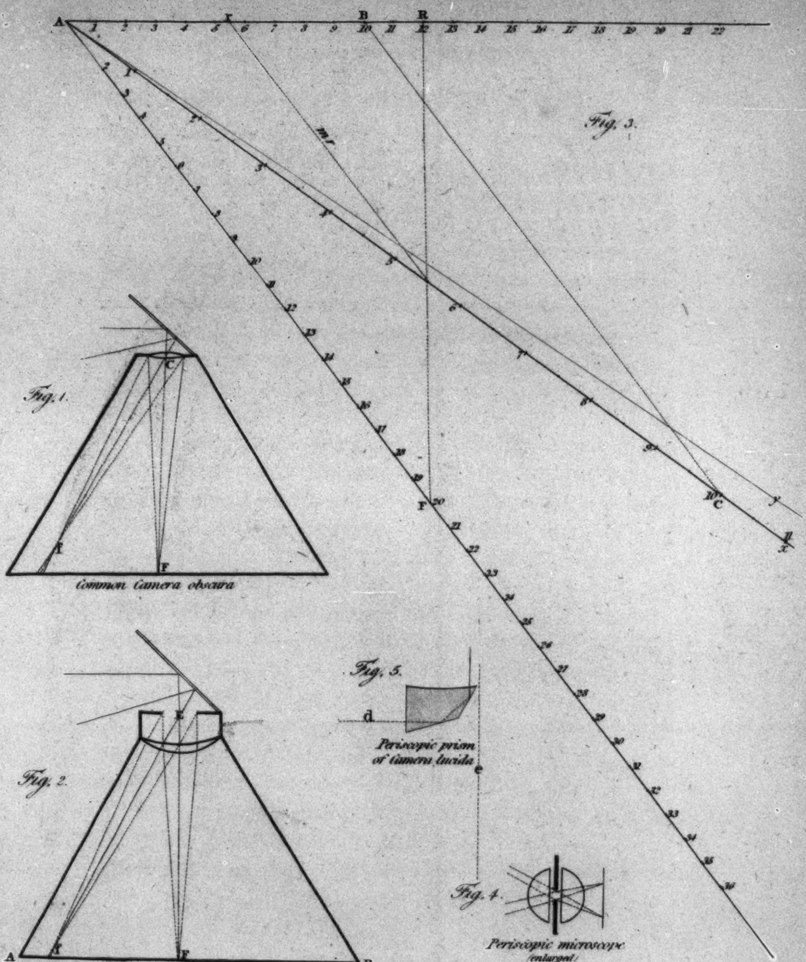

Fig. 3.

Fig. 1.

Common Camera obscura

Fig. 2.

Periscopic Camera obscura

Fig. 5.

Periscopic prism of Camera lucida

Fig. 4.

Periscopic microscope (enlarged)

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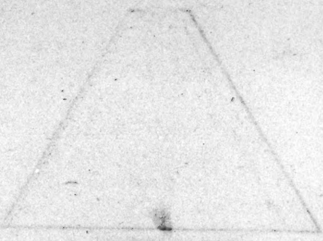
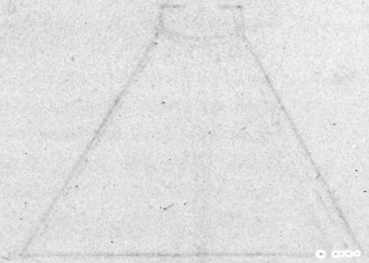
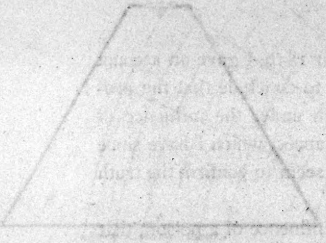
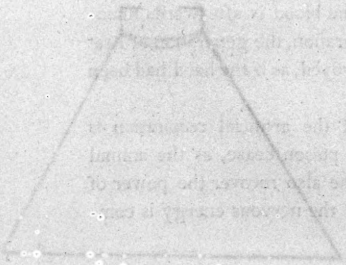




tinctly without the use of a concave glass, it may be assisted in that respect by a due degree of concavity given to either, or to both the transmitting surfaces of the prism. It is, however, to the upper surface alone that this concavity is given; for since the eye is then situated on the side toward the centre of curvature, it receives all the benefit that is proposed from the periscopic principles.





XIX. *Further Experiments and Observations on the influence of the Brain on the generation of Animal Heat. By* B. C. Brodie, *Esq. F. R. S. Communicated to the Society for promoting the knowledge of Animal Chemistry, and by them to the Royal Society.*

Read June 18, 1812.

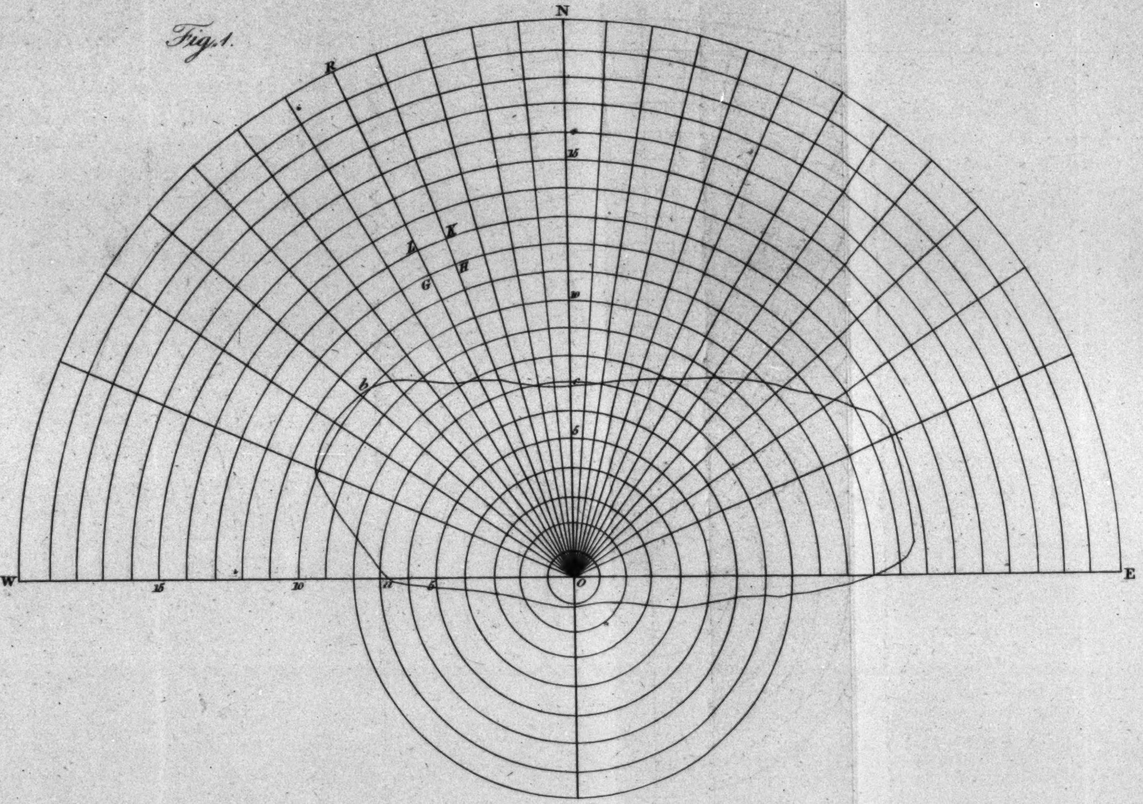
In the Croonian Lecture for the year 1810, I gave an account of some experiments, which led me to conclude that the production of animal heat is very much under the influence of the nervous system. Some circumstances, which I have since met with, illustrate this subject, and seem to confirm the truth of my former conclusions.

In an animal, which is under the influence of a poison, that operates by disturbing the functions of the brain, in proportion as the sensibility becomes impaired, so is the power of generating heat impaired also.

If an animal is apparently dead from a poison of this description, and the circulation of the blood is afterwards maintained by means of artificial respiration, the generation of heat is found to be as completely destroyed, as if the head had been actually removed.

Under these circumstances, if the artificial respiration is kept up until the effects of the poison cease, as the animal recovers his sensibility, so does he also recover the power of generating heat; but it is not till the nervous energy is com-

Fig. 1.



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pletely restored, that heat is produced in sufficient quantity to counteract the cold of the surrounding atmosphere.*

In the experiments formerly detailed, as well as in those just mentioned, I observed that the blood underwent the usual alteration of colour in the two systems of capillary vessels, while carbonic acid was evolved from the lungs at each expiration; and hence I was led to believe, that the respiratory function was performed nearly as under ordinary circumstances, and that the usual chemical changes were produced on the blood. It appeared, however, desirable to obtain some more accurate knowledge on this point, and I have therefore instituted a series of experiments, for the purpose of ascertaining the relative quantities of air consumed in breathing, by animals in a natural state, and by animals in which the brain has ceased to perform its office, and I now have the honour of communicating an account of these experiments to this Society.

It has been shewn, by Messrs. ALLEN and PEPYS, first,† that every cubic inch of carbonic acid requires exactly a cubic inch of oxygen gas for its formation; secondly,‡ that when respiration is performed by a warm-blooded animal in atmospheric air, the azote remains unaltered, and the carbonic acid exactly equals, volume for volume, the oxygen gas, which disappears.

There is therefore reason to believe, that the watery vapour, which escapes with the air in expiration, is not formed from the union of hydrogen with oxygen in the lungs, but that it is exhaled from the mucous membrane of the mouth and pharynx, resembling the watery exhalation which takes place from the peritonæum, or any other internal surface when exposed; and this conclusion appears to be fully confirmed by the experiments of M. Magendie, lately communicated to the National Institute of Paris.

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* The poison employed in this experiment should be the essential oil of almonds, or some other, the effects of which speedily subside. If the woorara is employed, so long a time elapses before the poison ceases to exert its influence, that it becomes necessary that the experiment should be made in a high temperature, otherwise the great loss of heat which takes place, is sufficient to prevent recovery.

† Phil. Trans. 1807, Part II.

‡ Phil. Trans. 1808, Part II. Ibid. 1809, Part II.

These circumstances are of importance in the present communication, which they render more simple, as they show, that, in order to ascertain the changes produced on the air in respiration, it is only necessary to find the quantity of carbonic acid given out from the lungs. This becomes an exact measure of the oxygen consumed, and the azote of the air and the watery vapour expired, need not be taken into the account.

For the purpose of examining the changes produced on the air, by animals breathing under the different circumstances abovementioned, I contrived the apparatus, which is represented in the annexed Plate.

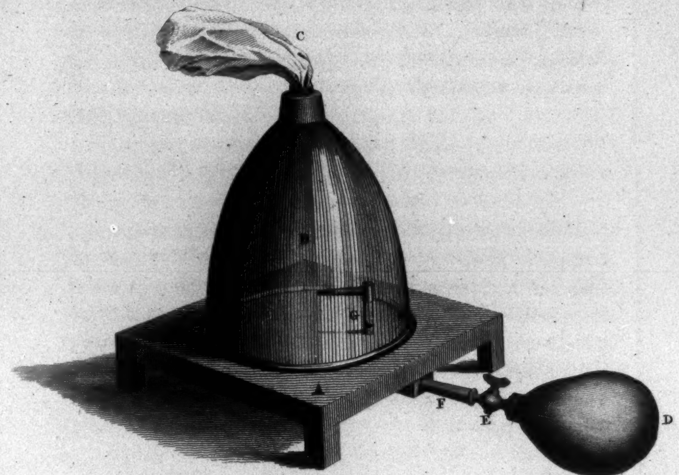
Description of the Apparatus.

A. Is a wooden stand, in which is a circular groove ¾ of an inch in depth, and the same in width.

B. Is a bell-glass, the rim of which is received in the circular groove of the wooden stand. In the upper part of the bell-glass is an opening, admitting a tube connected with the bladder C.

D. Is a bottle of elastic gum, having a brass stop-cock E connected with it.

F. Is a silver tube, of which one end is adapted to receive the tube of the stop-cock E, while the other extremity, making



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a right angle with the rest of the tube, passes through a hole in the wooden stand, and projects into the cavity of the bell-glass, where it makes a second turn also at a right angle, and becomes of a smaller diameter. In the upright part of the tube is an opening G.

The tubes are made perfectly air-tight, where connected with each other, and with the rest of the apparatus, and the circular groove is filled with quicksilver.

The capacity of the bell-glass, allowance being made for the rim, which is received in the groove with the quicksilver, is found to be 502 cubic inches. The capacity of the gum-bottle is 52 cubic inches, and in the calculations after the experiments 2 cubic inches have been allowed for the air contained in the different tubes, and for the small remains of air in the bladder after being nearly emptied by pressure.

Mode of using the Apparatus.

In order to ascertain the quantity of air consumed under ordinary circumstances, the animal was placed on the stand under the bell-glass, the bladder being emptied by pressure, and the gum-bottle being distended with atmospheric air. During the experiment, by pressing occasionally on the gum-bottle, the air was forced from it into the bell-glass. On removing the pressure, the gum-bottle became filled by its own elasticity with air from the bell-glass. Thus the air was kept in a state of agitation, and the dilatation of the bladder prevented the air being forced through the quicksilver under the edge of the bell-glass. At the end of the experiment, the gum-bottle was completely emptied by pressure, and allowed to be again filled with air from the bell-glass: this was repeated two or three times, and the air in the bottle was then preserved for examination. The proportion of carbonic acid being ascertianed, and the capacities of the different parts of the apparatus, and the space occupied by the animal being known, the total quantity of carbonic acid formed, and consequently of oxygen consumed, was easily estimated.

When the experiment was made on an animal, in whom the functions of the brain were destroyed, and in whom therefore voluntary respiration had ceased, the narrow extremity of the tube was inserted into an artificial opening in the trachea, and the animal being placed under the bell-glass, the lungs were inflated at regular intervals, by means of pressure made on the gum-bottle. The tube being smaller than the trachea, the greater portion of the air in expiration escaped by the side of the tube into the general cavity of the bell-glass, while the gum-bottle filled itself by its own elasticity with air through the opening G. At the end of the experiment, a portion of air was preserved for examination, and the quantity of carbonic acid was estimated in the way already described.

The animals employed in these experiments were of the same species, and nearly of the same size. Attention to these circumstances was judged necessary, that the results might be as conclusive as possible. The chemical examination of the air was made by agitating it in a graduated measure over quicksilver, with a watery solution of potash. My friend Mr. Brande gave me his assistance in this part of the present investigation, as he had done on many former occasions. It will be observed, that in estimating the proportion of carbonic acid, no allowance has been made for that contained in the atmospheric air; first, because the quantity is so small that the omission can occasion no material error; and secondly, because the object is to ascertain, not so much the absolute, as the relative quantities of carbonic acid evolved by animals breathing under different circumstances.

The experiments which I shall first notice, were made on the respiration of animals in a natural state.

Experiment 1. Thermometer 65°, barometer not noted.

A young rabbit was allowed to remain under the bell-glass during 30 minutes. The respired air at the end of this time was found to contain $\frac{1}{20}$ of carbonic acid.

It was ascertained that the rabbit occupied the space of 50 cubic inches.

The capacity of the bell-glass = 502 cubic inches.

That of the gum-bottle 52 cubic inches.

The air in the tubes and bladder = 2 cubic inches.

Then $\frac{502+52+2-50}{20} = \frac{506}{20} = 25.3.$

The rabbit therefore in 30 minutes gave out 25.3 cubic inches of carbonic acid, and consumed the same quantity of oxygen gas, which is at the rate of 50.6 in an hour.

Exp. 2. Thermometer 65°, barometer 30.1 inches.

A somewhat smaller rabbit was allowed to remain under the bell-glass during 30 minutes. The respired air contained $\frac{1}{18}$ of carbonic acid. The animal occupied the space of 48 cubic inches.

$$\frac{502+52+2-48}{18} = \frac{508}{18} = 28.22.$$

The carbonic acid evolved was therefore equal to 28.22 cubic inches in half an hour, which is at the rate of 56.44 cubic inches in an hour.

Exp. 3. Thermometer 64°, barometer 30.2 inches.

A young rabbit, occupying the space of 48 cubic inches, was allowed to remain under the bell-glass, during the same period as in the two former instances. The respired air contained $\frac{1}{18}$ of carbonic acid.

$$\frac{502 + 52 + 2 - 48}{18} = \frac{508}{18} = 28.22.$$

The results of this were therefore precisely the same as those of the last experiment.

These experiments were made with great care. The animals did not appear to suffer any inconvenience from their confinement, and their temperature was unaltered.

The next order of experiments were made for the purpose of ascertaining the quantity of air consumed by animals, in which the circulation of the blood was kept up by means of artificial respiration, after the brain had ceased to perform its functions.

Exp. 4. Thermometer 65°, barometer not noted.

Having procured two rabbits of the same size and colour, I divided the spinal marrow in the upper part of the neck of one of them. An opening was made in the trachea, and the lungs were inflated at first by means of a small pair of bellows. Two ligatures were passed round the neck, one in the upper, and the other in the lower part, behind the trachea. The ligatures were drawn tight, including every thing but the trachea; and the nerves, vessels, and other soft parts between them were divided with a bistoury. Eight minutes after the division of the spinal marrow, the thermometer in the rectum had sunk to 97°. The animal was placed under a bell-glass, and the lungs were inflated by pressing on the gum-bottle about

50 times in a minute. When this process had been continued for 30 minutes, a portion of air was preserved for examination. The heart was found acting regularly, but slowly, the thermometer in the rectum had fallen to 90°.

The second rabbit was killed by dividing the spinal marrow about the same time when the experiment was begun on the first rabbit. Being in the same temperature, the time was noted when the thermometer in the rectum had fallen to 97°, and it was placed under another bell-glass, that it might be as nearly as possible under the same circumstances with the first rabbit. At the end of 30 minutes, the thermometer in the rectum had fallen from 97 to 91.*

The air respired by the first rabbit contained $\frac{1}{25}$ of carbonic acid. The bulk of the rabbit was found = 50 cubic inches.

$$\frac{502 + 52 + 2 - 50}{25} = \frac{506}{25} = 20.24.$$

20.24 cubic inches of carbonic acid were therefore extricated in 30 minutes, which is at the rate of 40.48 cubic inches in an hour.

The carbonic acid given out in the same space of time was less than in the former experiments; but it is to be observed, first, that in consequence of the ligatures the extent of the circulation was diminished; secondly, that in this instance one of the ligatures accidentally slipped, and an ounce of blood was lost in the beginning of the experiment.

As it was desirable to avoid any circumstances, which might occasion a difference in the results, in the subsequent experiments I employed animals, which had been inoculated with the poison of woorara, or the essential oil of almonds; by which means, while the functions of the brain were completely destroyed, the extent of the circulation was undiminished, and all chance of accidental hæmorrhage was avoided.

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* In measuring the heat of the rectum in these experiments, care is necessary that the thermometer should always be introduced to exactly the same distance from the external parts, otherwise no positive conclusion can be drawn relative to the loss of heat, as the more internal parts retain their heat longer than the superficial.

Exp. 5. Thermometer 65°, barometer 29.8 inches.

Two rabbits were procured, each occupying the space of 45 cubic inches. They were both inoculated with the woorara poison.

The first rabbit was apparently dead in nine minutes after the application of the poison; but the heart continued to act. The lungs were inflated for about two minutes, by means of a pair of bellows, when the thermometer in the rectum was observed to stand at 98°. The animal was placed under the bell-glass, and artificial respiration was produced by means of pressure on the gum-bottle, as in the last experiment. At the end of 30 minutes, a portion of air was preserved for examination. The thermometer in the rectum had fallen to 91°. The heart still acted with regularity and strength.

The second rabbit died in a few minutes after the inoculation. The time was noted when the thermometer in the rectum had fallen to 98°, and he was placed under a bell-glass. At the end of 30 minutes, the thermometer in the rectum had fallen to 92°.

The air respired by the first rabbit contained $\frac{1}{10}$ of carbonic acid.

$\frac{502 + 52 + 2 - 45}{20} = \frac{511}{20} = 25.55$ cubic inches of carbonic acid evolved in 30 minutes, which is at the rate of 51.1 cubic inches in an hour.

Exp. 6. Thermometer 66°, barometer 30.1 inches.

Two rabbits, each occupying the space of 48 cubic inches, were inoculated with woorara.

In one of them, when apparently dead, the circulation was kept up by means of artificial respiration. He was placed in the apparatus under the bell-glass, and the lungs were inflated from 50 to 60 times in a minute. At this time the thermometer in the rectum stood at 97°. At the end of 35 minutes, a portion of air was preserved for examination. The thermometer had now fallen to 90°. The heart was still acting regularly.

The second rabbit was allowed to lie dead. When the thermometer in the rectum had fallen to 97°, he was placed under another bell-glass. At the end of 35 minutes, the thermometer had fallen to 90°.5.

The air respired by the first rabbit contained $\frac{1}{16}$ of carbonic acid.

$\frac{502 + 2 + 52 - 48}{16} = \frac{508}{16} = 31.75$ cubic inches of carbonic acid evolved in 35 minutes, which is at the rate of 54.43 cubic inches in an hour.

Exp. 7. Thermometer 60°, barometer 30.2 inches.

The experiment was repeated on a rabbit, which had been inoculated with the essential oil of almonds. When he was placed under the bell-glass, the thermometer in the rectum stood at 96°. In a few minutes he gave signs of sensibility, and made efforts to breathe; but as these efforts were at long intervals, the artificial respiration was continued. In half an hour he breathed spontaneously 40 times in a minute. The thermometer in the rectum had fallen to 90°.

The air being examined, was found to contain $\frac{1}{18}$ of carbonic acid.

The rabbit occupied the space of 47 cubic inches.

$\frac{502+52+2-47}{18} = \frac{509}{18} = 28.275$ cubic inches of carbonic acid evolved in 30 minutes, which is at the rate of 56.55 cubic inches in an hour.

The animal lay as if in a state of profound sleep. At the end of two hours and twenty minutes, from the time of the poison being applied, the thermometer in the rectum had fallen to 79°, and he was again apparently dead; but the heart still continued acting, though feebly, and its action was kept up for 30 minutes longer by means of artificial breathing, when the thermometer had fallen to 76°. The carbonic acid evolved during these last 30 minutes, amounted to nearly 13. cubic inches.

From the precautions with which these experiments were made, I am induced to hope that there can be no material error in their results. They appear to warrant the conclusion, that in an animal in which the brain has ceased to exercise its functions, although respiration continues to be performed, and the circulation of the blood is kept up to the natural standard, although the usual changes in the sensible qualities of the blood take place in the two capillary systems, and the same quantity of carbonic acid is formed as under ordinary circumstances; no heat is generated, and (in consequence of the cold air thrown into the lungs) the animal cools more rapidly than one which is actually dead.

It is a circumstance deserving of notice, that so large a quantity of air should be consumed by the blood passing through the lungs, when the functions of the brain, and those of the organs dependant on it, are suspended. Perhaps it is not unreasonable to suppose, that by pursuing this line of investigation we may be enabled to arrive at some more precise knowledge respecting the nature of respiration, and the purposes which it answers in the animal economy. It would however be foreign to the plan of the present communication to enter into any speculations on this subject, and I shall therefore only remark, that the influence of the nervous system does not appear to be necessary to the production of the chemical changes, which the blood undergoes in consequence of exposure to the air in the lungs.*

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* This conclusion is directly contrary to that deduced by M. DUPUYTREN, from a series of curious experiments, made with a view to ascertain the effects which follow the division of the nerves of the par vagum, and it is an object of some importance in the present investigation, to ascertain in what manner the apparently opposite facts, observed by M. DUPUYTREN and myself, are to be reconciled with each other.

It was observed by this physiologist, that in an animal, in which both the nerves of the par vagum are divided, the blood returned from the lungs has a darker colour than natural, and that the animals, on whom this operation is performed, die sooner or later with symptoms of asphyxia, notwithstanding the air continues to enter the lungs; and hence he concludes, that the changes which are produced on the blood in respiration are not the result of a mere chemical process, but are dependent on the nervous influence, and cease to take place when the communication between the lungs and the brain is destroyed.

M. PROVENÇAL, in prosecuting this inquiry, ascertained that the animals subjected to this experiment give out less carbonic acid than before.

M. BLAINVILLE observed, that the frequency of the inspirations is much diminished; and M. DUMAS restored the scarlet colour of the arterial blood by artificially inflating the lungs, and from these and other circumstances, he has arrived at conclusions very different from those of M. DUPUYTREN.

My own observations exactly correspond with those of M M. DUMAS and BLAINVILLE. After the nerves of the par vagum are divided, a less quantity of carbonic

The facts now, as well as those formerly adduced, go far towards proving, that the temperature of warm-blooded animals is considerably under the influence of the nervous system; but what is the nature of the connection between them? whether is the brain directly or indirectly necessary to the production of heat? these are questions to which no answers can be given, except such as are purely hypothetical. At present we must be content with the knowledge of the insulated fact: future observations may, perhaps, enable us to refer it to some more general principle.

We have evidence, that, when the brain ceases to exercise its functions, although those of the heart and lungs continue to be performed, the animal loses the power of generating heat. It would, however, be absurd to argue from this fact, that the chemical changes of the blood in the lungs are in no way necessary to the production of heat, since we know of no instance in which it continues to take place after respiration has ceased.

It must be owned, that this part of physiology still presents an ample field for investigation.

Of opinions sanctioned by the names of BLACK, LAPLACE, LAVOISIER, and CRAWFORD, it is proper to speak with caution

acid is evolved, the inspirations are much diminished in frequency, and the blood in the arteries of the general system assumes a darker hue; but its natural colour may be restored by artificially inflating the lungs, so as to furnish a greater supply of air to the blood circulating through them.

We may suppose, that, on the division of these nerves, the sensibility of the lungs is either extremely impaired, or altogether destroyed, so that the animal does not feel the same desire to draw in fresh air: in consequence his inspirations become less frequent than natural, and hence arise the phenomena produced by this operation.

and respect, but without trespassing on these feelings, I may be allowed to say, that it does not appear to me that any of the theories hitherto proposed afford a very satisfactory explanation of the source of animal heat.

Where so many and such various chemical processes are going on, as in the living body, are we justified in selecting any one of these for the purpose of explaining the production of heat?

To the original theory of Dr. BLACK, there is this unanswerable objection, that the temperature of the lungs is not greater than that of the rest of the system. To this objection, the ingenious and beautiful theory of Dr. CRAWFORD is not open; but still it is founded on the same basis with that of Dr. BLACK, "the conversion of oxygen into carbonic acid in the lungs," and hence it appears to be difficult to reconcile either of them with the results of the experiments which have been related.

It may perhaps be urged, that as in these experiments the secretions had nearly, if not entirely ceased, it is probable that the other changes, which take place in the capillary vessels had ceased also, and that although the action of the air on the blood might have been the same as under ordinary circumstances, there might not have been the same alteration in the specific heat of this fluid, as it flowed from the arteries into the veins. But, on this supposition, if the theory of Dr. CRAWFORD be admitted as correct, there must have been a gradual, but enormous accumulation of latent heat in the blood, which we cannot suppose to have taken place without its nature having been entirely altered. If the blood undergoes the usual change in the capillary system of the pulmonary, it is probable that it must undergo the usual change in the capillary system of the greater circulation also, since these changes are obviously dependent on and connected with each other. The blood in the aorta and pulmonary veins was not more florid, and that in the vena cava and pulmonary artery was not less dark-coloured than under ordinary circumstances. We may moreover remark, that the most copious secretions in the whole body are those of the insensible perspiration from the skin, and of the watery vapour from the mouth and fauces, and the effect of these must be to lower rather than to raise the animal temperature. Under other circumstances also the diminution of the secretions is not observed to be attended with a diminution of heat. On the contrary, in the hot fit of a fever, when the scanty dark-coloured urine, dry skin, and parched mouth indicate that scarcely any secretions are taking place, the temperature of the body is raised above the natural standard, to which it falls when the constitution returns to its natural state, and the secretions are restored.

It has been observed, by a distinguished chemist, that "the experiments to determine the specific heat of the blood are of so very delicate a nature, that it is difficult to receive them with perfect confidence."* The experiments of Dr. Crawford for this purpose were necessarily made on blood out of the body, and at rest. Now, when blood is taken from the vessels, it immediately undergoes a remarkable chemical change, separating into a solid and a fluid part. This separation is not complete for some time; but whoever takes the pains to make observations on the subject, can hardly doubt that it begins to take place immediately on the blood being drawn. Can experiments on the blood, under these circumstances, lead to any very satisfactory conclusions, respecting the specific heat of blood circulating in the vessels of the body? The diluting the blood with large quantities of water, as proposed by Dr. CRAWFORD, does not altogether remove the objection, for this only retards, it does not prevent coagulation, and some time must, at any rate, elapse, while the blood is flowing and the quantity is being measured, during which the separation of its solid and fluid parts will have begun to take place.

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* Thomson's History of the Royal Society, p. 129.

More might be said on this subject; but I feel anxious to avoid, as much as possible, controversial discussion. It is my wish not to advance opinions, but simply to state some facts, which I have met with in the course of my physiological investigations. These facts, I am willing to hope, possess some value; and they may perhaps lead to the developement of other facts of much greater importance. Physiology is yet in its infant state. It embraces a great number and variety of phenomena, and of these it is very difficult to obtain an accurate and satisfactory knowledge; but it is not unreasonable to expect, that by the successive labours of individuals, and the faithful register of their observations, it may at last be enabled to assume the form of a more perfect science.

XX. *On the different Structures and Situations of the Solvent Glands in the digestive Organs of Birds, according to the nature of their Food and particular Modes of Life.* By Everard Home, *Esq. F. R. S.*

Read June 18, 1812.

The solvent glands in birds are larger and more distinct from the other parts of the digestive organs than in the class mammalia, which has enabled me to ascertain many circumstances respecting their structure, not to be determined by examining the stomachs of quadrupeds. An account of these is contained in the present paper.

To make the following descriptions more clear and distinct, I shall divide the digestive organs of birds, whether they live on animal or vegetable food, into four parts. The first, is the dilatation of the œsophagus, which forms a reservoir for the food, and which is called the crop. The second, is the part into which the ducts of the solvent glands open, which, I shall call the cardiac cavity. The third, is the cavity embraced by the digastric muscle, or gizzard. The fourth, is the space between the opening of the gizzard, and beginning of the duodenum, which I shall call the pyloric cavity, although in some instances it appears scarcely to deserve that name.

The solvent glands in the whole of the extensive genus falco of Linnæus, are cylindrical bodies with very small canals, a villous internal surface, and thick coats, open at one end, closed, and rounded off at the other; they are placed on the outside of the membrane which lines the cardiac cavity, they lie parallel to one another, and nearly at right angles to the membrane through which they open, the closed end being slightly turned upwards, so as to make the orifice the most depending part. In the golden eagle, (the Falco Chrysaëtos *Linn.*) and the sea eagle, (Falco ossifragus), they form, altogether, a broad compact belt: but in the hawk, (Falco Nisus) this belt is slightly divided into four distinct portions; immediately below these glands the cavity becomes wider, and is inclosed in a digastric muscle of weak power, with a flat tendon on each side. The internal surface of this cavity, which is the gizzard, is soft and vascular.*

In all birds that live on animal food, the solvent glands appear to have a similar structure to that which has been just described, only differing in size and situation. The following are the most material differences which I have met with respecting their situation.

In the Soland goose, (Pelecanus bassanus,) these glands are rather larger than in the eagle, but are placed in the dilated part of the cavity of the gizzard, forming a complete belt of great breadth, consequently are extremely numerous. Their situation and appearance is shewn in the annexed drawing.

In the heron, (Ardea cinerea), the solvent glands are in the same situation as in the Soland goose; they are thinly scattered, and do not form a complete belt, being more numerous on the anterior and posterior surfaces. A ball of fish bones held together by mucus, was found in the cavity of the gizzard.

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* An engraving of this appearance is given in the Philosophical Transactions for the year 1807, p. 178, pl. xii.

In the cormorant, (Pelecanus Carbo), the situation of the solvent glands is the same as in the Soland goose, but they only form two circular spots, one anterior, the other posterior.*

In all these birds, the inner membrane of the gizzard is soft and smooth, but that portion which covers the solvent glands, has a more spongy or villous appearance; and this part is found to secrete a mucus which the other parts do not. This fact appears to be ascertained by the following circumstances: on examining the gizzard of a cormorant that died in consequence of an inflammation in the œsophagus, which had been communicated to the internal membrane of the gizzard, a viscid mucus was found upon the surface covering the solvent glands, and this was not met with in any other part, so that the mucus had been evidently secreted there, and was afterwards coagulated by the liquor of the solvent glands poured upon it, coagulation being the first process which takes place in the act of digestion. This explains the circumstance of ascarides being frequently found enveloped in mucus in this part of the cormorant's gizzard, the mucus on which they feed being secreted in consequence of the irritation they produce on the membrane. In the same manner the flukes in the biliary ducts of the sheep, increase the secretion of the bile by irritating these canals, and then feed on it.

It is generally believed that mucus is secreted by surfaces as well as by glandular structure, but I know of no evidence that the mucus of the stomach comes under this description, except what is now stated.

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* An engraving of this gizzard is given in the Phil. Trans. for the year 1807, pl. x. p. 178.

In birds that live upon fish, and sea insects with crustaceous coverings, as the sea gull, (Larus canus), the gizzard has a horny cuticular lining, and the solvent glands are in the same situation as in the genus falco. In the gizzard were found the scales of small fishes. The appearance of the solvent glands and gizzard are shewn in the annexed engraving.

In those birds that live on land insects, some of whose coverings are soft, others hard, there is a difference in the structure of the digestive organs from what has been described. The solvent glands are placed in a triangular form in the cardiac cavity, which is very large, and immediately under it is a small gizzard with a horny lining. Of this kind is the woodpecker, (Picus minor). A representation of the parts in this bird is annexed.

There is still another variety in the structure of these organs.

In the little auk, the (Alca Alle), the solvent glands are spread over a greater extent of surface than in any other bird that lives on animal food, and the form of the digestive organs is peculiar to itself. The cardiac cavity appears to be a direct continuation of the œsophagus, distinguished from it by the termination of the cuticular lining, and the appearance of the solvent glands. This cavity is continued down with very gradual enlargement below the liver, and is then bent up to the right side, and terminates in a gizzard: when the cavity is laid open, the solvent glands are seen at its upper part, every where surrounding it, but lower down they lie principally on the posterior surface, and where it is bent upwards towards the right side, they are entirely wanting. The gizzard has a portion of its anterior and posterior surfaces opposite each other, covered with a horny cuticle. These appearances are shewn in the annexed drawing.

This peculiar formation of the digestive organs of the little auk, appears to be fitted for economizing the food; which may be rendered more necessary in a bird, that spends a portion of the year in the frozen regions of the North, where supplies of nourishment must be very precarious.

In birds that live principally on vegetable food, the solvent glands have a different structure, according to the substances the birds are intended to feed upon, and vary a good deal in situation, according to the habits of life. The following are the most remarkable instances of such difference, both with respect to structure and situation.

In the pigeon (Columba domestica), their situation is the same as in the genus falco, but their size is small, the external orifices large, and the coats thin, so that they resemble the glands in the English heron, but having larger cavities.

In the swan, the (Anas Cygnus), the solvent glands appear to be cylinders, as in the genus falco, but are not straight, bending upon one another in a direction obliquely upwards; their internal surface is not villous as in the genus falco, but rather broken and irregular.

In the goose (Anas Anser), the solvent glands have the same situation as in the swan, and resemble them in their external appearance, but when laid open the sides are found to be cellular.

In the common fowl (Phasianus Gallus), these glands are made up of four small short processes uniting in a middle tube, which opens externally by one orifice.

In the turkey (the Meleagris Gallipavo), the solvent glands consist of four small processes, which diverge from one another in opposite directions.

In a species of parrot, (the Psittacus æstivus), the cardiac cavity is unusually large and long, and the solvent glands are spread over a considerable portion of its surface, the gizzard is very small. Its appearance is represented in the annexed engraving.

In many large birds that only walk and run, their wings being too small to enable them to fly, the digestive organs are formed in many respects differently from those of other birds.

In the cassowary, (Casuarius Emeu), the solvent glands are situated between the crop and gizzard, as in many other birds, but this part is dilated into a large cavity, and separated from the gizzard by an oblique muscular valve; in this cavity the food may be retained for some time, but cannot be triturated there, since the stones and other hard bodies swallowed, will readily force a passage into the gizzard.

I have not had an opportunity of examining the solvent glands in the cassowary, and therefore can say nothing respecting their structure from my own observation.

In the American ostrich, (the Rhea americana), the solvent glands are fewer in number than in other birds. They only occupy a small portion of a circular form, on the posterior side of the cardiac cavity; this however is compensated by the complex structure of which they are composed. To each gland there is one common orifice; when the cavity to which it leads is laid open, three smaller orifices are exposed, each of which communicates with five or six processes like the fingers of a glove. The structure is similar to that of the solvent glands of the beaver, among quadrupeds.

The cardiac cavity, in which the solvent glands are situated, is dilated to a large size, as in the cassowary, and there is a similar muscular valve separating it from the gizzard. The digastric muscle is weak; but the fibres of which it is composed, and the tendons between the two bellies of the muscle, are beautifully distinct. The appearance of these parts is shewn in the annexed drawing.

In the African ostrich, (the Struthio Camelus,) the solvent glands are unusually numerous, similar in structure to those of the American species; the space in which they are situated is not only dilated into a cavity, but is continued down below the liver, and then bent up upon itself towards the right side, where it terminates in a strong gizzard nearly at the same height as the beginning of the cardiac cavity.

This cardiac cavity of such unusual length and uncommon form is lined with a strong cuticle, except upon the left side where the solvent glands are placed, extending from the top to the bottom, and about four inches in breadth.

The gizzard is unusually small; the grinding surfaces do not admit of being separated to a great distance from one another, and on one side there are two grooves, and two corresponding ridges on the other. Beyond the cavity of the gizzard is an oval aperture with six ridges covered with cuticle to prevent any thing passing out of the gizzard till it is reduced to a small form. These appearances are shewn in the annexed drawing.

In this bird the reverse takes place of what was mentioned to happen in the cassowary and American ostrich, for the stones and other hard bodies swallowed by these birds, must, from their weight, force a way into the gizzard, which is a large cavity adapted to receive them; but here all such substances must remain in the cardiac cavity, both from its being the most depending part, and from the cavity of the gizzard being too small to admit of their entering it.

The cardiac cavity contained stones of various sizes, pieces of iron, and halfpence; but between the grinding surfaces of the gizzard, there were only broken glass beads of different colours, and hard gravel mixed with the food.

In taking a review of the structure of the digestive organs of the cassowary, the American, and African ostrich, whose mode of progressive motion is the same, we find their organs very differently circumstanced with respect to the means of œconomising their food.

The cassowary and American ostrich differ from birds that fly, in having the solvent glands placed in a cavity of unusual size and the muscular structure of the gizzard uncommonly weak; the mode of progressive motion, which is a kind of run, producing so much agitation between the stones and the food, as to render a stronger muscular action unnecessary.

In the cassowary there appear to be no considerations of œconomy in the management of the food in the process of digestion, the solvent glands are less complex than in the ostrich, as is avowed by those who have examined them,* the food has a free passage from the gizzard into the intestines, which are unusually wide and short, so that its passage through them is very rapid, and is rendered still more so by the stones of a large size employed in the gizzard passing out at the anus. This I learnt from Sir JOSEPH BANKS, who was present at the Cape of Good Hope when one of these birds, to his great astonishment, voided nearly half a bucket full of stones.

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* Vide PERRAULT's Comp. Anat. 1676.

In the American ostrich there will be less waste of the food than in the cassowary, as the solvent glands are of a more complex structure, as there is a less ready outlet from the gizzard, and as the intestines are longer and have a variety of convolutions.

In the African ostrich the means of œconomising the food are greater than in other birds; the glands have the same structure as in the American species, are more numerous, are spread over a larger surface, there is a more extensive cavity in which the substances it feeds upon are triturated; and beyond this, a grooved gizzard for the more accurate breaking down of the food. The intestines also are longer and more varied in their course.

All these provisions of nature fit this bird to live in the sandy deserts, of which it is the natural inhabitant; and are not bestowed upon the others that live in countries where food is more abundant.

It is a curious circumstance that the situation of the solvent glands, the shape of the cardiac cavity and position of the gizzard in the alca alle among carnivorous birds is nearly the same as that of the African ostrich among birds that live principally on vegetable food.

EXPLANATION OF THE PLATES.

PLATE XI.

The gizzard of the Soland goose laid open to show the situation and appearance of the solvent glands. The engraving is of the natural size.

PLATE XII.

Shows the form and internal structure of the cardiac cavity and gizzard in the wood-pecker, the sea-gull, and the little auk, all of the natural size.

Fig. 1. The external appearance of the cardiac cavity and gizzard in the wood-pecker.

2. The internal appearance of the cardiac cavity and gizzard of the wood-pecker.

3. The internal appearance of the cardiac cavity and the cavity of the gizzard in the sea-gull.

4. The internal appearance of the cardiac cavity and gizzard of the little auk.

PLATE XIII.

Two views of the digestive organs of the parrot of the natural size.

Fig. 1. An external view of the crop, cardiac cavity, and gizzard.

2. An internal view of the same parts.

PLATE XIV.

An external view of the cardiac cavity and gizzard of the American ostrich, one-fourth of the natural size.

Plate XV.

An internal view of the cardiac cavity [illegible] gizzard of the American ostrich, on the same scale as [illegible] plate.

The orifices of the solvent glands [illegible] conspicuous in the cardiac cavity.

Plate XVI.

An external view of the cardiac cavit[illegible]ard of the African ostrich, one-sixth of the natural s[illegible]

Plate XVII.

Represents the internal view of the A[illegible] gizzard, and also a series of solvent glands b[illegible]erent birds.

Fig. 1. The internal surface of the cardi[illegible] gizzard of the African ostrich. On the same sc[illegible] last plate.

Fig. 2. A series of solvent glands [illegible]erent appearances which the[illegible]

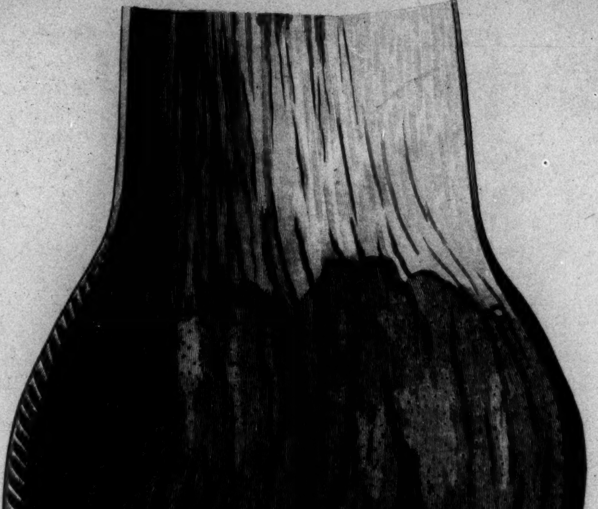

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Fig. 1.

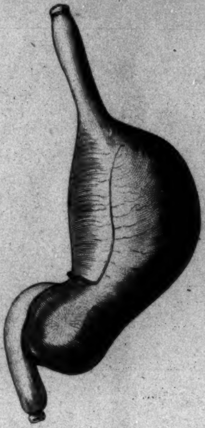


Fig. 2.

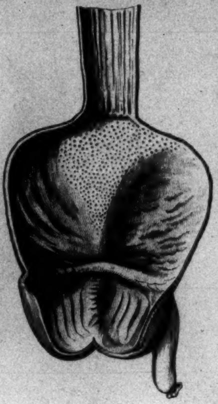


Fig. 3.

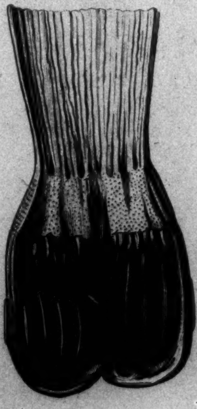
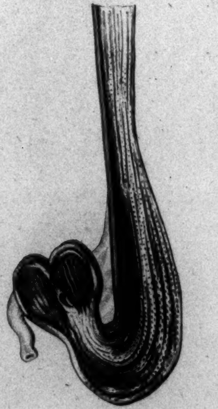


Fig. 4.



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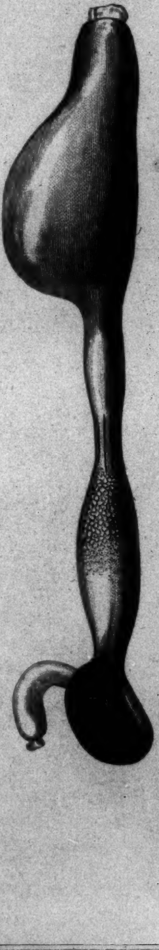

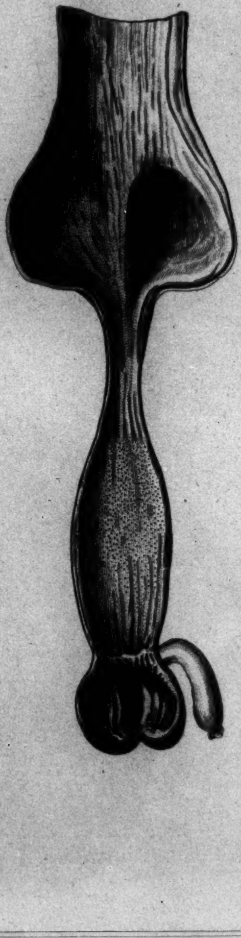

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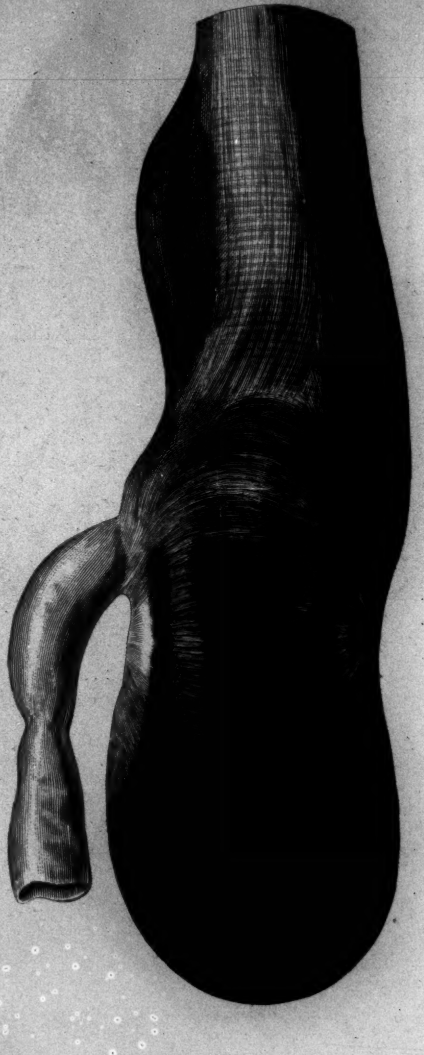

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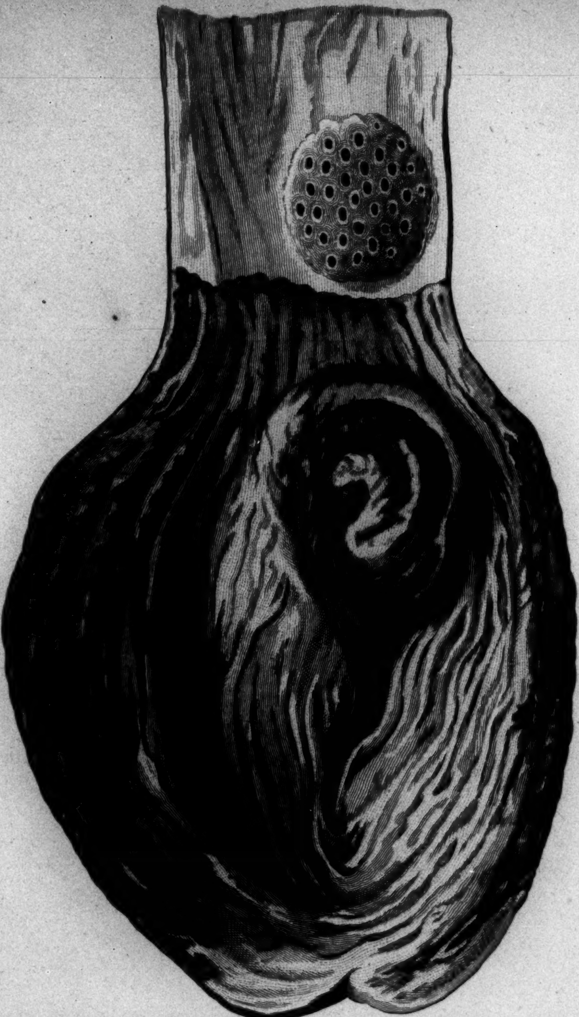
Philos. Trans. MDCCCXII. *Plate* XV. *p.* 404.



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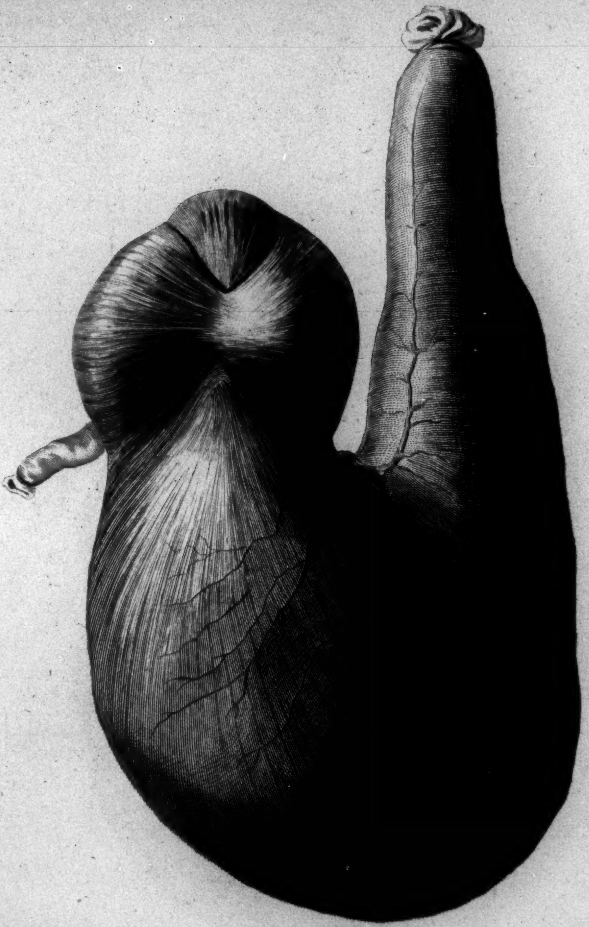

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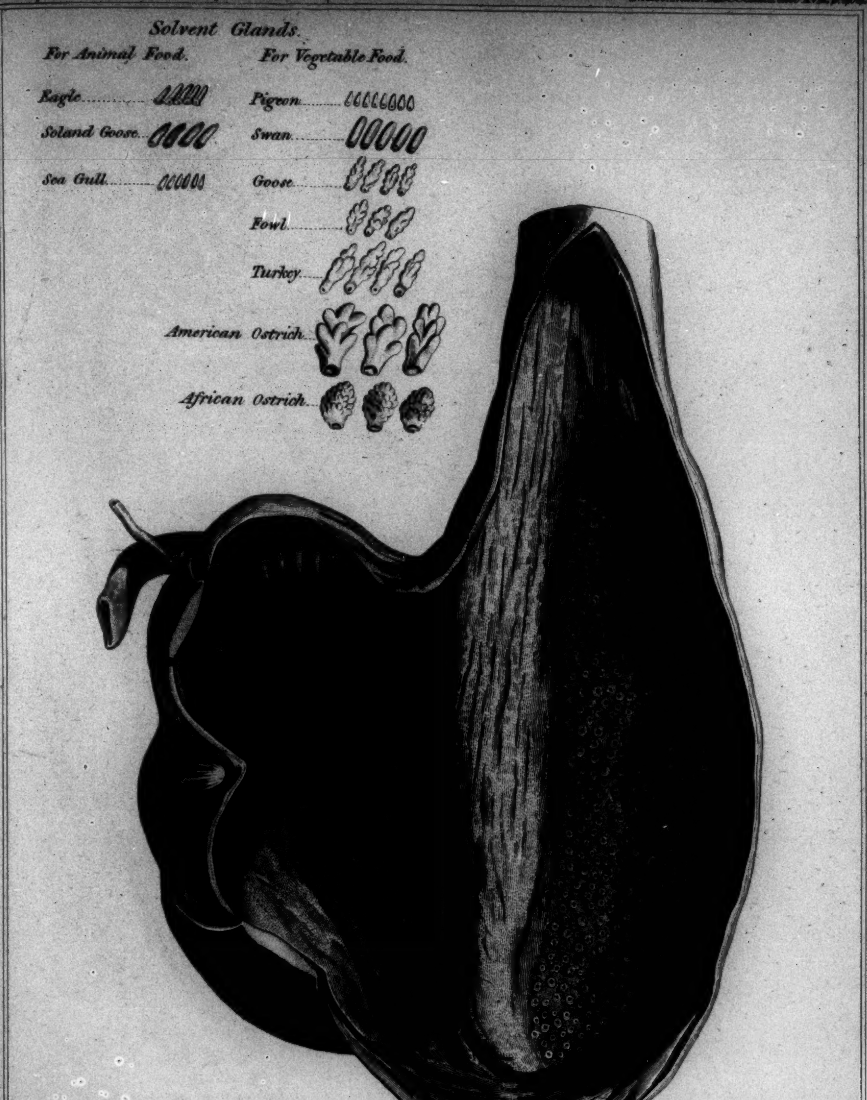

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XXII. *On some Combinations of Phosphorus and Sulphur, and on some other Subjects of Chemical Inquiry. By Sir* Humphry Davy, *Knt. LL. D. Sec. R. S.*

Read June 18, 1812.

1. *Introduction.*

In this paper, I shall do myself the honour of laying before the Society, the results of some experiments on phosphorus and sulphur, which establish the existence of some new compounds, and which offer decided evidences in favour of an idea that has been for some time prevalent amongst many enlightened chemists, and which I have defended in former papers published in the Philosophical Transactions; namely, that bodies unite in definite proportions, and that there is a relation between the quantities in which the same element unites with different elements.

I shall not enter into a minute detail of the methods of experimenting that I employed; I shall confine myself to general statements of the facts. The common manipulations of chemistry are now too well known to require any new illustrations: and to dwell upon familiar operations, would be to occupy unnecessarily and tediously the time of this learned body.

2. *Of some Combinations of Phosphorus.*

In a paper read before the Royal Society in 1810, I have described the mutual action of phosphorus and oxymuriatic gas, or chlorine. I have noticed two compounds which appear to be distinct and peculiar bodies, formed by the union of the gas and the inflammable substance. One is solid, white, and crystalline in its appearance; easily volatile, and capable of forming a fixed infusible substance by uniting with ammonia. The other is fluid, limpid as water, and, as I have since found, of specific gravity 1.45; it produces dense fumes by acting upon the water of the atmosphere, and when exposed to the atmosphere gradually disappears, leaving no residuum.

The composition of the white sublimate is very easily ascertained by synthetical experiments, such as I have described on a former occasion in the Transactions. By employing chlorine dried by muriate of lime, in great excess, and making the experiments in exhausted vessels, and admitting solution of chlorine to ascertain the quantity of gas absorbed, I have ascertained that 3 grains of phosphorus unite with about 20 grains of chlorine to form the sublimate.

If the phosphorus be in great excess in the experiment of its combustion in chlorine, some of the liquor is formed with the sublimate; but to obtain it in considerable quantities, phosphorus should be passed in vapour through heated powdered corrosive sublimate. A bent glass tube may be used for the process, and the liquor condensed in a cold vessel connected with the tube.

I have not been able to determine its composition by synthetical experiments; but by pouring it gradually into water, suffering the water to become cool after each addition of the liquor, and then precipitating the solution by solution of nitrate of silver, I have ascertained the quantity of chlorine and of phosphorus it contains. 13.6 grains, treated in this way, afforded 43 grains of horn silver.

It is evident from this analysis, compared with the result of the synthetical experiments on the sublimate, that the quantity of phosphorus being the same, the sublimate contains double as much chlorine as the liquor.

When phosphorus is heated in the liquor, a portion is dissolved, and it then when exposed to the atmosphere leaves a film of phosphorus, which when the liquor is thrown on paper usually inflames: a substance of this kind was first procured by M M. Gay Lussac and Thenard, by distilling phophorus and calomel together; and it may be produced in the experiment with corrosive sublimate, if sufficient heat be used to sublime the phosphorus, or if there be not an excess of the corrosive sublimate. I have made no experiments in order to ascertain the quantity of phosphorus the liquor will dissolve.

When the white sublimate is made to act upon water, it dissolves in it producing much heat. The solution evaporated affords a thick liquid, which is a solution of pure phosphoric acid, or a hydrat of phosphoric acid.

When the liquor is treated with water in the same way, it furnishes likewise a thick fluid of the consistence of syrup, which crystallizes slowly by cooling, and forms transparent parallelopipedons.

This substance has very singular properties: when it is heated pretty strongly in the air, it takes fire and burns brilliantly, emitting at the same time globules of gas, that inflame at the surface of the liquid. This substance may be called *hydrophosphorous* acid; for it consists of pure phosphorous acid and water. This is proved by the action of ammoniacal gas upon it; when it is heated in contact with ammonia, water is expelled and phosphite of ammonia formed; and it is likewise shewn by the results of its decomposition in close vessels, which are phosphoric acid and a peculiar compound of phosphorus and hydrogen.

Ten parts in weight of the crystalline acid I found produced about 8.5 parts of solid phosphoric acid, and the elastic product must of course have formed the remainder of the weight, allowing for a small quantity of the substance not decomposed.

The peculiar gas is not spontaneously inflammable; but explodes when mixed with air, and heated to a temperature rather below 212°.

Its specific gravity appeared from an experiment in which a small quantity of it only was weighed, to be to that of air nearly as 87 to 100. Water absorbed about one-eighth of its volume of this gas. Its smell was disagreeable, but not nearly so fetid as that of common phosphuretted hydrogene.

When it was detonated with oxygene, it was found that three of it in volume absorbed more than five in volume of oxygene, and a little phosphorus was precipitated.

When potassium was heated in contact with it, its volume increased rapidly till it became double, and then no further effect was produced. The potassium was partly converted into a substance having all the characters of phosphuret of potassium; and the residual gas absorbed the same quantity of oxygene by detonation as pure hydrogene. When sulphur was sublimed in the gas over mercury, the volume was likewise doubled; a compound of phosphorus and sulphur was formed, and the elastic fluid produced had all the characters of sulphuretted hydrogene.

It appears from these experiments, that the peculiar gas consists of 4.5 of hydrogene in weight to 22.5 phosphorus; and its composition being known, it is easy to determine the composition of the hydrophosphorous acid, and likewise the quantity of oxygene required by a given quantity of phosphorous acid to be converted into phosphoric acid; for, for every volume of gas disengaged, a volume of oxygene must have been fixed in the phosphoric acid.

And calculating for 174 grains, 30 parts of oxygene must be fixed in the 150 parts of phosphoric acid, and 20 parts of phosphorus disengaged in combination with 4 parts of hydrogene; and on the idea of representing the proportions in which bodies combine by numbers, if hydrogene be considered as unity and water as composed of two proportions of hydrogene, 2, and one of oxygene 15,[*] phosphorus will be represented by 20.

When the compounds of chlorine and phosphorus are acted on by a small quantity of water, muriatic acid gas is disengaged with violent ebullition, the water is decomposed, and it is evident that for every volume of hydrogene disengaged in combination with the chlorine, half a volume of oxygene must be combined with the phosphorus; and the products of the mutual decomposition of water, and the phosphoric compounds of chlorine are merely the phosphoric acid from the sublimate and the phosphorous acid from the liquor, and muriatic acid gas; so that the quantity of phosphorus being the same, it is evident that phosphoric acid must contain twice as much oxygene as phosphorous acid, which harmonizes with the results of the decomposition of hydrophosphorous acid. For supposing water to be composed of two proportions of hydrogene, and one of oxygene, and the number representing it 17; then 174 parts of hydrophosphorous acid must consist of two proportions; 34 parts of water, and four proportions of phosphorous acid, containing 80 of phosphorus and 60 of oxygene; and three proportions of phosphoric acid must be formed, containing three proportions of phosphorus 60, and six proportions of oxygene 90, making 150.

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[*] Supposing 100 cubical inches of the gas to weigh 27 grains.——27 — 4.5 the weight of 200 cubical inches of hydrogene = 22.5 grains.

[†] This mode of estimation is the same as that I have adopted on a former occasion, except that the number representing oxygene is doubled to avoid a fractional part.

It is scarcely possible to imagine more perfect demonstrations of the laws of definite combination, than those furnished in the mutual action of water and the phosphoric compounds. No products are formed except the new combinations; neither oxygene, hydrogene, chlorine, nor phosphorus is disengaged, and therefore the ratio in which any two of them combine being known, the ratios in which the rest combine, in these cases, may be determined by calculation.

I converted phosphorus into phosphoric acid, by burning it in a great excess of oxygene gas over mercury in a curved glass tube, and heated the product strongly. I found in several processes of this kind, that for every grain of phosphorus consumed, four cubical inches and a half of oxygene gas were absorbed; which gives phosphoric acid as composed of 20 of phosphorus to 30.6 of oxygene; a result as near as can be expected to the results of the experiments on the sublimate and the hydrophosphorous acid.

Unless the product of the combustion of phosphorus is strongly heated in oxygene, the quantity of oxygene absorbed is less, so that it is probable that phosphorous acid is formed, as well as phosphoric acid.

Phosphorous acid is usually described, in chemical authors, as a fluid body, and as formed by the slow combustion of phosphorus in the air; but the liquid so procured is, I find, a solution of a mixture of phosphorous and phosphoric acids. And the vapour arising from phosphorus in the air at common temperatures, is a combination of phosphorous acid and the aqueous vapour in the air, and is not, I find, perceived in air artificially dried.

In this case, the phosphorus becomes covered with a white film, which appears to be pure phosphorous acid, and it soon ceases to shine.

A solid acid, volatile at a moderate degree of heat, may be produced by burning phosphorus in very rare air, and this seems to be phosphorous acid free from water; but some phosphoric acid, and some yellow oxide of phosphorus, are always formed at the same time.

The peculiar gas differs exceedingly from phosphoretted hydrogene formed by the action of earths and alkalies and phosphorus upon water; for this last gas is spontaneously inflammable, and its specific gravity is seldom more than half as great, and it does not afford more than 1.5 its volume of hydrogene when decomposed by potassium; it differs in its qualities in different cases, and probably consists of different mixtures of hydrogene with a peculiar gas, consisting of 2 parts of hydrogene and

20 of phosphorus; or it must contain several proportions of hydrogene to one of phosphorus.

I venture to propose the name *hydrophosphoric* gas for the new gas; and according to the principles of nomenclature, I have proposed in the last Bakerian lecture, the liquor containing 20 of phosphorus to 67 of chlorine may be called *phosphorane*, and the sublimate *phosphorana*.

3. *Of some Combinations of Sulphur.*

I have shewn, in a paper published in the Philosophical Transactions for 1810, that sulphuretted hydrogene is formed by the solution of sulphur in hydrogene, and I have supposed that sulphureous acid, in like manner, is constituted by a solution of sulphur in oxygene. There is always a little condensation of volume in experiments on the combustion of sulphur in oxygene; but this may fairly be attributed to some hydrogene loosely combined in the sulphur; and to the production of a little sulphuric acid by the mutual action of hydrogene, oxygene, and sulphur.

It is only necessary, if these data be allowed, to know the difference between the specific gravity of sulphureous acid gas and oxygene, and sulphuretted hydrogene and hydrogene, to determine their composition.

In the Philosophical Transactions for 1810, page 254, I have somewhat under-rated the weights of sulphuretted hydrogene and sulphureous acid gasses: for I have since found, that the cubical inch measures, employed for ascertaining the volumes of gas weighed, were not correct. From experiments which I think may be depended upon, as the weights of the gasses were merely compared with those of equal volumes of common air, I found that 100 cubical inches of sulphureous acid gas weighed 68 grains at mean temperature and pressure, and 100 cubical inches of sulphuretted hydrogene 36.5 grains, and the last result agrees very nearly with one given by MM. GAY LUSSAC and THENARD, and one gained by my brother Mr. JOHN DAVY.

If 34, the weight of 100 cubical inches of oxygene gas, be subtracted from 68, it will appear that sulphureous acid consists of equal weights of sulphur and oxygene, an estimation which agrees very nearly with one given by M. BERZELIUS; and if 2.27, the weight of 100 cubical inches of hydrogene be subtracted from 36.5, the remainder 34.23 will be the quantity of sulphur in the gas; and the number representing sulphur may be stated as 30; and sulphureous acid as composed of one proportion of sulphur 30, and two of oxygene 30; and sulphuretted hydrogene as composed of one proportion of sulphur, and two of hydrogene.

From the experiments of MM. GAY LUSSAC, it appears that sulphuric acid decomposed by heat affords one volume of oxygene to two of sulphureous acid: from this it would appear to be composed of one proportion of sulphur to three of oxygene. I have endeavoured, in several trials by common heat and by electricity, to combine sulphureous acid gas with oxygene, so as to form a sulphuric acid free from water, but without success; and it is probable, that three proportions of oxygene cannot be combined with one proportion of sulphur, except by the intermedium of water. Mr. DALTON has supposed, that there is a solid sulphuric acid formed by the action of sulphureous acid gas upon nitrous acid gas. But I find, that when dried sulphureous acid gas and nitrous acid gas are mixed together, there is no action; but by introducing the vapour of water, they form together a solid crystalline hydrat; which when thrown into water gives off nitrous gas, and forms a solution of sulphuric acid.

I have referred, in the Philosophical Transactions, to the combination of chlorine and sulphur. I have been able to form no compound of these bodies, which does not deposit sulphur by the action of water. When sulphur is saturated with chlorine, as in Dr. THOMSON's sulphuretted liquor, it appears to contain, from my experiments, only 67 of chlorine to 30 of sulphur.

4. *Some general Observations.*

It is a fact worthy of notice, that phosphoric and sulphuric acids should contain the same quantity of oxygene to the same quantity of inflammable matter; and yet that the oxygene should be combined in them, with such different degrees of affinity. Phosphorous acid has a great tendency to unite with oxygene, and absorbs it even from water: and sulphureous acid can only retain it when water is present.

The relation of water to the composition of many bodies has already occupied the attention of some distinguished chemists, and is well worthy of being further studied; most of the substances obtained by precipitation from aqueous solutions are, I find, compounds of water.

Thus zircona, magnesia, silica, when precipitated and dried at 212° still contain definite proportions of water. And many of the substances which have been considered as metallic oxides, that I have examined, obtained from solutions, agree in this respect; and their colours and other properties are materially influenced by this combined water.

I shall give an instance. The substance which has been called the white oxide of manganese is a compound of water and the protoxide of manganese, and when heated strongly, it gives off its water and becomes a dark olive oxide.

It has been often suspected, that the contraction of volume produced in the pure earths by heat, is owing to the expulsion of water combined with them. The following fact seems to confirm this suspicion, and offers a curious phenomenon.

Zircona, precipitated from its solution in muriatic acid by an alkali, and dried at a temperature below 300°, appears as a white powder, so soft as not to scratch glass. When heated to 700° or 800°, water is suddenly expelled from it, and notwithstanding the quantity of vapour formed, it becomes at the moment red hot. After the process, it is found harsh to the feel, has gained a tint of gray, its parts cohere together, and it is become so hard as to scratch quartz.

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